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The Nonsymmetric Kaluza–Klein (Jordan–Thiry) Theory in the Electromagnetic Case

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We present the nonsymmetric Kaluza–Klein and Jordan–Thiry theories as interesting propositions of physics in higher dimensions. We consider the five-dimensional (electromagnetic) case. The work is devoted to a five-dimensional unification of the NGT (nonsymmetric theory of gravitation), electromagnetism, and scalar forces in a Jordan–Thiry manner. We find "interference effects" between gravitational and electromagnetic fields which appear to be due to the skew-symmetric part of the metric. Our unification, called the nonsymmetric Jordan–Thiry theory, becomes the classical Jordan–Thiry theory if the skew-symmetric part of the metric is zero. It becomes the classical Kaluza–Klein theory if the scalar field $\rho=1$ (Kaluza's Ansatz). We also deal with material sources in the nonsymmetric Kaluza–Klein theory for the electromagnetic case. We consider phenomenological sources with a nonzero fermion current, a nonzero electric current, and a nonzero spin density tensor. From the Palatini variational principle we find equations for the gravitational and electromagnetic fields. We also consider the geodetic equations in the theory and the equation of motion for charged test particles. We consider some numerical predictions of the nonsymmetric Kaluza–Klein theory with nonzero (and with zero) material sources. We prove that they do not contradict any experimental data for the solar system and on the surface of a neutron star. We deal also with spin sources in the nonsymmetric Kaluza–Klein theory. We find an exact, static, spherically symmetric solution in the nonsymmetric Kaluza–Klein theory in the electromagnetic case. This solution has the remarkable property of describing "mass without mass" and "charge without charge." We examine its properties and a physical interpretation. We consider a linear version of the theory, finding the electromagnetic Lagrangian up to the second order of approximation with respect to $h_{\mu\nu}=g_{\mu\nu}-\eta_{\mu\nu}$. We prove that in the zeroth and first orders of approximation there is no skewon–photon interaction. We deal also with the Lagrangian for the scalar field (connected to the "gravitational constant"). We prove that in the zeroth and first orders of approximation the Lagrangian vanishes.

INTRODUCTION

The aim of this paper is to construct the Kaluza–Klein (Jordan–Thiry) analogue of Einstein's geometry on the electromagnetic bundle. In other

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words, it will be a five-dimensional unification of the NGT (nonsymmetric theory of gravitation), classical Maxwell electromagnetism, and scalar forces connected to the gravitational constant (as in scalar–tensor theories of gravitation). Our unification uses a nonsymmetric metrization of fiber bundles.

The electromagnetic bundle means a principal fiber bundle over a space-time E with a structural group U(1). The connection defined on this bundle is called an electromagnetic connection.

Roughly speaking, in general relativity, mass curves space-time. In NGT, mass and fermion charge (fermion number) curve and twist space-time. In the classical Kaluza–Klein theory, mass curves space-time and electric charge curves the fifth dimension. In the nonsymmetric Kaluza–Klein theory, mass and fermion number curves and twist space-time, and electric charge curves and twists the fifth dimension.

NGT is based on three fundamental geometrical quantities: two connections $\bar{\Gamma}^{\alpha}{}_{\beta\gamma}$ and $\bar{W}^{\alpha}{}_{\beta\gamma}$ and the nonsymmetric metric $g_{\alpha\beta}$. This nonsymmetric metric is equivalent to the existence of two geometrical objects defined on space-time: the symmetric metric tensor

$$\bar{g} = g_{\langle\alpha\beta\rangle}\bar{\theta}^{\alpha} \otimes \bar{\theta}^{\beta}$$

and the two-form

$$\underline{g} = g_{[\mu\nu]}\bar{\theta}^{\mu} \wedge \bar{\theta}^{\nu}$$

In the general theory of relativity, we have only one connection with vanishing torsion and a symmetric metric on space-time. Thus, we have only $\bar{\Gamma}$ and $\bar{g}$. Of course, in NGT, connections $\bar{\Gamma}$ and $\bar{W}$ are interrelated and have nonvanishing torsion.

The classical Kaluza–Klein approach was based on the following ideas.[(1,2),2]

On space-time we have Riemannian geometry based on the metric tensor $\underline{g}$, and we have general relativity with the local coordinate invariance principle. Simultaneously we have a principal fiber bundle over space-time with the structural group U(1). The connection on this bundle describes the electromagnetic field. We have also the local gauge invariance principle for an electromagnetic field.

The local coordinate invariance principle and the local gauge invariance principle seem to be two important principles of invariance. The Kaluza–Klein theory unifies these two concepts and reduces them to the first, the local coordinate invariance principle, but in a five-dimensional world.

The basic idea is very simple. On the gauge group U(1) we have a bi-invariant symmetric tensor. The tensor plays the role of a metric in the Lie

[2] In this paper, references are cited as superscript numerals in parentheses.

algebra of the gauge group U(1) which is simply R (real numbers). We can choose as this tensor the number (-1).

On the electromagnetic bundle we have the natural distribution of horizontal spaces induced by the connection.

The metric tensor $\bar{\mathbf{g}}$ acts on space-time.

We can divide every tangent vector to the electromagnetic bundle in only one way (the connection is established) into two parts: horizontal and vertical. The horizontal part we can project onto space-time and the vertical one, due to the connection, onto the Lie algebra of the gauge group (i.e., onto R). Thus, we have (symmetric) metrization of the fiber bundle. We can "measure" independently the length of both parts by two (symmetric) metric tensors and after this add these two results. Having the principal fiber bundle metrized in this way, we introduce a linear connection on the bundle which is compatible in some sense with the metric. The simplest solution is to suppose that this connection is the Levi-Civita connection. This was done in the five-dimensional Kaluza–Klein theory[1,2] If we calculate the Ricci curvature scalar for this connection, we get the sum of the Ricci curvature scalar on space-time and the electromagnetic Lagrangian.

Introducing the scalar field in a Jordan–Thiry manner, we get a Lagrangian for a scalar field. However, this term can be removed as a four-divergence from the Lagrangian density. This, means it does not propagate in the five-dimensional Riemannian case. Thus, the five-dimensional Jordan–Thiry (Kaluza–Klein) theory does not offer any "interference effects" between gravity and electromagnetism. This forces us to abandon the Riemannian geometry defined on the electromagnetic bundle and to use a different connection. In our case it is a five-dimensional generalization of the geometry from Einstein's unified field theory[3-6] (in the Kaufman version[4,5]) defined on the electromagnetic bundle. This geometry is bi-invariant with respect to the action of the group U(1). It defines the Einstein–Kaufman U(1) structure.

This theory, the nonsymmetric Kaluza–Klein theory, unifies the coordinate invariance principle from NGT and the local gauge invariance principle from electrodynamics.

Following the ideas concerning the geometry of the Kaluza–Klein theory described above, it is necessary to find the nonsymmetric metrization of the electromagnetic bundle over space-time. The existence of a nonsymmetric metric on the fiber bundle is equivalent to the existence of two geometrical objects: $\bar{\gamma}$ and $\underline{\gamma}$. The first, $\bar{\gamma}$, is a symmetric bi-invariant tensor that is the same as in the classical Kaluza–Klein (Jordan–Thiry) theory, and the second, $\underline{\gamma}$, is a 2-form on the fiber bundle. (For classical results see Refs. 7–61.)

Following the basic idea of the previous construction, it is necessary to choose a 2-form on the gauge group U(1). This form is zero on U(1), because every 2-form on U(1) is zero. This means that in the electromagnetic case $\gamma = \pi^*(\mathbf{g})$, where π^* is the pullback of π (the natural projection on the electromagnetic bundle). We can also introduce the scalar field in the Jordan–Thiry manner.

In this new version of the Jordan–Thiry theory we get the following new results. We get "interference effects" between electromagnetic and gravitational fields,(18,19,25) i.e.:

1. A new term in the electromagnetic Lagrangian

$$\frac{1}{4\pi}(\mathbf{g}^{[\mu\nu]}F_{\mu\nu})^2$$

2. The existence of an electromagnetic polarization of the vacuum $M_{\alpha\beta}$ with the interpretation as a torsion in the fifth dimension.
3. An additional term for the Lorentz force term in the equation of motion for a test particle

$$\left(\frac{q}{m_0}\right)\mathbf{g}^{[\gamma\alpha]}H_{\gamma\beta}u^\beta$$

where q is the charge of the test particle and m_0 its rest mass.
4. A new energy-momentum tensor $T^{em}_{\mu\nu}$ for an electromagnetic field with zero trace.
5. The source for the electromagnetic field—the conserved current j_α.

All of these effects vanish if the metric of space-time becomes symmetric.

We get in the Moffat–Ricci curvature scalar on a five-dimensional manifold $\underline{P}$ the Lagrangian of the scalar field Ψ,

$$\mathscr{L}_{\text{scal}} = \mathbf{g}^{[\nu\mu]}\mathbf{g}_{\delta\nu}\tilde{\mathbf{g}}^{(\alpha\delta)}\Psi_{,\mu}\Psi_{,\alpha}$$

and this field is connected to the effective gravitational constant by $K = e^{-3\Psi}$ (K is the gravitational "constant"). The trace of the energy-momentum tensor for this field is not zero. This suggests that Ψ is massive and has Yukawa-type behavior. This indicates that Ψ has short range and the theory does not violate the equivalence principle. Furthermore, the gravitational "constant" K does not change at long distances. This statement also supports the masslike term in the equation for Ψ

$$-24\pi\, e^{-3\Psi}\mathscr{L}_{\text{em}}$$

where

$$\mathscr{L}_{\text{em}} = \frac{1}{8\pi}[2(\mathbf{g}^{[\mu\nu]}F_{\mu\nu})^2 - H^{\alpha\mu}F_{\alpha\mu}]$$

is the Lagrangian for the electromagnetic field in our theory.

We also get a scalar-force term in the equation of motion for a charged test particle moving in the gravitational and electromagnetic fields:

$$-\frac{1}{4}\left(\frac{q}{m_0}\right)^2 \tilde{\mathbf{g}}^{(\alpha\beta)} e^{2\Psi}\Psi_{,\beta}$$

This force is of short range if Ψ is of short range. In our theory all of these additional effects (in comparison to the classical Kaluza–Klein theory) vanish if the metric of space-time becomes symmetric. First of all, Ψ does not propagate. It is easy to see that

$$\mathscr{L}_{\text{scal}}=0$$

for

$$\mathbf{g}_{[\mu\nu]}=0$$

and because of this, the additional term in the equation of motion for a test particle also vanishes. It is of course important to find significant physical consequences of the "interference effects" present in the nonsymmetric Kaluza–Klein (Jordan–Thiry) theory. The best way to achieve this is to find an exact solution of the full field equations. We find an exact solution of the field equations in the static, spherically symmetric case in the form suggested in Section 6 of ref. 18. Even in this, the simplest, case we get the following interesting results[30,41]:

1. The electric field is nonsingular at $r=0$ and has Coulomb-like behavior for large r. This is similar to the situation in Born–Infeld electrodynamics.[87] Thus, there is a maximal value of the electric field.
2. Asymptotically (for large r) the full solution behaves like the charged Reissner–Nördström type solution in NGT.
3. The Newtonian mass (mass seen at infinity) equals the total energy of the solution and is constructed from an electric charge Q and from a fermion charge l.
4. The energy distribution is not singular everywhere. This means that the solution describes a bounded system of electromagnetic and gravitational fields.
5. There is no singularity at $r=0$ in the function $\alpha=\mathbf{g}_{11}$; that is, $\mathbf{g}_{11}(r=0)=1$.
6. The only singularities at $r=0$ are in $\omega=\mathbf{g}_{[14]}=l^2/r^2$ and in a factor $(1+l^4/r^4)$ in the function $\gamma=\mathbf{g}_{44}$. There is also the usual singularity in the determinant of the full nonsymmetric tensor $\sqrt{-\mathbf{g}}=r^2\sin\theta$ at $r=0$.

7. The charge distribution is nonsingular.
8. For sufficiently large charge Q there exist one or two event horizons, just as in the Reissner–Nördström solution to the Einstein–Maxwell equations. Sufficiently large charge in the present case means sufficiently large Newtonian mass as well.

This solution is interesting as a classical model of a charged particle constructed from gravitational and electromagnetic fields. If we suppose that the Newtonian mass of our solutions is the mass of an electron, we get a relationship between the classical radius of an electron and the parameter l from NGT. The most fascinating aspect of our solution is that it describes "mass without mass" and "charge without charge" in the following sense. At the origin, $r=0$ (or anywhere) there are no Coulomb-like or Newton-like first- and second-order poles with charge and mass as residues. This is true for the metric and for the electric field.

Let us make some remarks on differences between the nonsymmetric Kaluza–Klein and Jordan–Thiry theories. In the nonsymmetric Kaluza–Klein theory there is an Ansatz $\rho=1$ $[\gamma_{55}(x)=-1]$. This condition seems to be quite arbitrary and because of this we consider a more general case called Jordan–Thiry theory where $\gamma_{55}(x)=-\rho^2(x)$ is a dynamical field.

Moreover, the detailed examination of the geodetic equations (for a curve Γ) in both cases reveals the following. If $\rho=\text{const}$, the geodetic equations possess an integral of motion

$$g_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}=\text{const} \tag{$*$}$$

or $\gamma(\text{hor}(u(\tau)), \text{hor}(u(\tau)))=\text{const}$ on $\Gamma\subset P$, which allows us to maintain an initial normalization of the four-velocity for a test particle. Horizontally is understood in the sense of the electromagnetic connection. In the case with $\gamma_{55}(x)\neq\text{const}$ (i.e., $\rho\neq 1$) this is not possible in general. We discuss this problem in Sections 9, 12, and 17. For this the condition $\gamma_{55}(x)=\text{const}$ does not seem to be an Ansatz in the theory but rather a conclusion from equation $(*)$.

This paper is organized as follows. In Section 1 we give some elements of the geometry. The second section describes the nonsymmetric tensor on a Lie group. The third gives a formulation of the nonsymmetric metrization of the principal bundle. In Section 4 we formulate the nonsymmetric Jordan–Thiry theory. We calculate connections $\omega^A{}_B$ and $W^A{}_B$ on the five-dimensional manifold which are analogous to connections $\bar{\omega}^\alpha{}_\beta$ and $\bar{W}^\alpha{}_\beta$ from NGT and Einstein–Kaufman theory. In Section 5 we write the geodetic equation on P (nonsymmetrically metrized electromagnetic bundle) and we find a new correction to the Lorentz force term. Section 6 is devoted to the calculation of the 2-forms of torsion and the curvature for the connection $\omega^A{}_B$. After this we write the curvature tensor for $\omega^A{}_B$ and its contractions. Using the obtained results, we calculate the Moffat–Ricci tensor and the Moffat–Ricci curvature scalar for the connection $W^A{}_B$. In Section 7 we deal with conformal transformation of $g_{\mu\nu}$ and a scalar field. In Section 8 we define the Palatini variational principle for the Moffat–Ricci curvature scalar $R(W)$. We get field equations for gravitational and electromagnetic fields. We discuss and interpret our results and point out all differences between the classical and the nonsymmetric Jordan–Thiry (Kaluza–Klein) theories. We write down all "interference effects" between gravitational and electromagnetic fields which appear in our theory. In Section 9 we deal in detail with an equation of motion for a test particle. In Section 10 we introduce material external sources into the nonsymmetric Kaluza–Klein theory. Section 11 is devoted to the spin sources in our theory. In Section 12 we examine geodetic equations in the case of nonzero external sources. Section 13 is devoted to numerical predictions of the theory. In Section 14 we deal with spin sources in the weak-field approximation. Section 15 is devoted to the linearization procedure in the nonsymmetric Jordan–Thiry (Kaluza–Klein) theory. In Section 16 we consider the equation of motion for a test particle in the linear approximation. Section 17 is devoted to the geodetic equations in the general case ($\rho \neq$ const) and to geodetic deviation equations. In Sections 18–22 we consider a stationary, spherically symmetric field in the nonsymmetric Kaluza–Klein theory. We find the exact solution of the field equation and examine its properties. In the Appendix we give some additional details of our calculations.

1. ELEMENTS OF GEOMETRY

In this section we introduce notations and define the geometric quantities used in the paper. We use a smooth principal fiber bundle $\underline{P}$, which includes in its definition the following list of differentiable manifolds and smooth maps:

A total (bundle) space $\underline{P}$.

A base space E; in our case it is a space-time.

A projection $\pi: \underline{P} \to E$.

A map $\Phi: P \times G \to \underline{P}$ defining the action of G on $\underline{P}$; if $a, b \in G$ and $\varepsilon \in G$ is the unit element, then $\Phi(a) \circ \Phi(b) = \Phi(ba)$, $\Phi(\varepsilon) = \mathrm{id}$, and $\Phi(a)p = \Phi(p, a) = R_a p = pa$; moreover, $\pi \circ \Phi(a) = \pi$. Here ω is a 1-form of a connection on $\underline{P}$ with values in the Lie algebra of the group G.

Let $\Phi'(a)$ be the tangent map to $\Phi(a)$ and $\Phi^*(a)$ be contragradient to $\Phi(a)$ at the point a. The form ω is a form of Ad type, i.e.,

$$\Phi^*(a)\omega = \mathrm{Ad}_{a^{-1}}\omega \tag{1.1}$$

where $\mathrm{Ad}_a \in GL(\mathrm{g})$ is the tangent map to the internal automorphism of the group G (i.e., it is an adjoint representation of a group G)

$$\mathrm{ad}_a(b) = aba^{-1}$$

Due to the form ω, we can introduce the distribution field of linear elements H_r, $r \in \underline{P}$, where $H_r \subset T_r(\underline{P})$ is a subspace of the space tangent to $\underline{P}$ at a point r and

$$v \in H_r \Leftrightarrow \omega(v) = 0 \tag{1.2}$$

We have

$$T_r(\underline{P}) = V_r \otimes H_r \tag{1.3}$$

where H_r is called the subspace of horizontal vectors and V_r that of vertical vectors. For vertical vectors $v \in V_r$ we have $\pi'(v) = 0$. This means that v is tangent to fibers. Let us define

$$v = \mathrm{hor}(v) + \mathrm{ver}(v), \qquad \mathrm{hor}(v) \in H_r, \qquad \mathrm{ver}(v) \in V_r \tag{1.4}$$

It is well known that the distribution H_r is equivalent to the choice of the connection ω. We can reproduce the connection form ω demanding that $\pi'_{|H_r}\colon H_r \to T_{\pi(r)}(E)$ is a vector space isomorphism ($\dim H_r = \dim E = 4$), $H_{\Phi(r,g)} = \Phi'(g)H_r$ [$T_{\pi(r)}(E)$ is a tangent space to space-time E at a point $\pi(r)$]. We use the operation hor for forms, i.e.,

$$(\mathrm{hor}\,\beta)(X, Y) = \beta(\mathrm{hor}\,X, \mathrm{hor}\,Y) \tag{1.5}$$

where $X, Y \in T_r(\underline{P})$. The 2-form of curvature of the connection ω is

$$\Omega = \mathrm{hor}\,d\omega \tag{1.6}$$

It is also a form of Ad type like ω. The 2-form Ω obeys the structural Cartan equation

$$\Omega = d\omega + \tfrac{1}{2}[\omega, \omega] \tag{1.7}$$

where $[\omega, \omega](X, Y) = [\omega(X), \omega(Y)]$. Bianchi's identity for ω is

$$\mathrm{hor}\,d\Omega = 0 \tag{1.8}$$

For the principal fiber bundle we use the following convenient scheme (Figure 1A). The map $e: U \to \underline{P}$, $U \subset E$ (U open), so that $e \circ \pi = \mathrm{id}_U$, is called a local section. From the physical point of view this means choosing

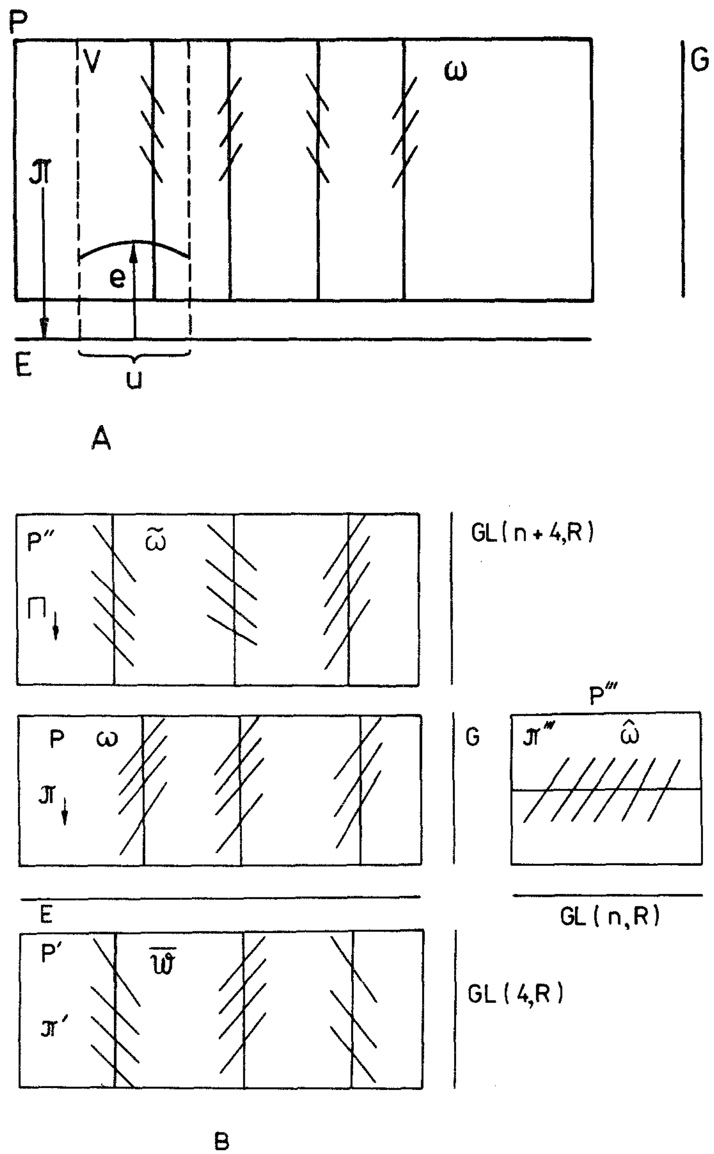

Fig. 1. (A) The principal fiber bundle $\underline{P}$. (B) Principal fiber bundles $\underline{P}$, P', P'', and P'''. Here P''' is the principal fiber bundle of frames over G.

the gauge. Thus,

$$e^*\omega = e^*(\omega^a X_a) = A^a{}_\mu \bar{\theta}^\mu X_a = A$$
$$e^*\Omega = e^*(\Omega^a X_a) = \tfrac{1}{2} F^a_{\mu\nu} \bar{\theta}^\mu \wedge \bar{\theta}^\nu X_a \tag{1.9}$$

We also introduce the notation

$$\Omega^a = \tfrac{1}{2} H^a_{\mu\nu} \theta^\mu \wedge \theta^\nu \tag{1.10}$$

where $\theta^\mu = \pi^*(\bar{\theta}^\mu)$ and

$$F^a_{\mu\nu} = \partial_\mu A^a{}_\nu - \partial_\nu A^a{}_\mu + C^a_{bc} A^b_\mu A^c_\nu$$

X_a ($a = 1, 2, \ldots, \dim G = n$) are generators of the Lie algebra $\mathfrak{g}$ of the group G and

$$[X_a, X_b] = C^c{}_{ab} X_c$$

Analogously we can introduce a second local section $f: U \to P$, and corresponding to it $\bar{A} = f^*\omega$, $\bar{F} = f^*\Omega$. For every $x \in U \subset E$ there is an element $g(x) \in G$ such that $f(x) = e(x)g(x) = R_{g(x)}e(x) = \Phi(e(x), g(x))$. Due to equation (1.1) and an analogous formula for Ω, one gets $\bar{A} = \mathrm{Ad}_{g^{-1}} A + g^{-1}\, dg$ and $\bar{F} = \mathrm{Ad}_{g^{-1}} F$. These formulas give a geometrical meaning of gauge transformation.

In this paper we use also a linear connection on manifolds $\underline{P}$ and E using the formalism of differential forms. So the basic quantity is a 1-form of the connection $\omega^A{}_B$. This is an R-valued (coefficient) connection form and it is referred to the principal fiber bundle of frames with $\underline{P}$ or E as a base. The 2-form of curvature is

$$\Omega^A{}_B = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B \tag{1.11}$$

and the 2-form of torsion

$$\Theta^A = D\theta^A \tag{1.12}$$

where θ^A are basic forms, and D means the exterior covariant derivative with respect to $\omega^A{}_B$. The following relations define the interrelation between our symbols and the generally used ones:

$$\omega^A{}_B = \Gamma^A{}_{BC} \theta^C$$
$$\Theta^A = \tfrac{1}{2} Q^A{}_{BC} \theta^B \wedge \theta^C \tag{1.13}$$
$$\Omega^A{}_E = \tfrac{1}{2} R^A{}_{BCD} \theta^C \wedge \theta^D$$

where $\Gamma^A{}_{BC}$ are coefficients of the connection (they do not have to be symmetric in indices B and C), $R^A{}_{BCD}$ is the tensor of curvature, and $Q^A{}_{BC}$ is the tensor of torsion. Covariant exterior differentiation with respect to $\omega^A{}_B$ is given by

$$D\Xi^A = d\Xi^A + \omega^A{}_C \wedge \Xi^C$$
$$D\Sigma^A{}_B = d\Sigma^A{}_B + \omega^A{}_C \wedge \Sigma^C{}_B - \omega^C{}_B \wedge \Sigma^A{}_C \tag{1.14}$$

The forms of curvature $\Omega^A{}_B$ and torsion Θ^A obey Bianchi identities

$$D\Omega^A{}_B = 0$$
$$F\Theta^A = \Omega^A{}_B \wedge \theta^B \tag{1.15}$$

In the paper we use also Einstein's + and − differentiations for the nonsymmetric metric tensor g_{AB},

$$Dg_{A+b-} = Dg_{AB} - g_{AD}Q^D{}_{BC}\theta^C \tag{1.16}$$

where D is the covariant exterior derivative with respect to $\omega^A{}_B$ and $Q^D{}_{BC}$ is the tensor of torsion for $\omega^A{}_B$. In a homolonomic system of coordinates we easily get

$$Dg_{A+B-} = g_{A+B-;C}\theta^C = (g_{AB,C} - g_{DB}\Gamma^D{}_{AC} - g_{AD}\Gamma^D{}_{CB})\theta^C \tag{1.17}$$

All quantities introduced in this section and their precise definitions can be found in Refs. 51 and 59–61.

Finally let us connect the general formalism of the principal fiber bundle with the formalism of a linear connection on E or $\underline{P}$.

Let M be an m-dimensional pseudo-Riemannian manifold with metric $\mathbf{g}$ of arbitrary signature. Let $T(M)$ be the tangent bundle and $O(M, \mathbf{g})$ the principal fiber bundle of frames (orthonormal frames) over M. The structure group of $O(M, \mathbf{g})$ is the group $Gl(m, \mathbb{R})$ or the subgroup of $Gl(m, \mathbb{R})$, $O(m-p, p)$, which leaves the metric invariant. Let Π be the projection of $O(M, \mathbf{g})$ onto M. Let X be a tangent vector at a point x in $O(M, \mathbf{g})$. The canonical or soldering form $\tilde{\theta}$ is an R^m-valued form on $O(M, \mathbf{g})$ whose Ath component $\tilde{\theta}^A$ at x of X is the Ath component of $\Pi'(X)$ in the frame x. The connection form $\tilde{\omega} = \omega^A{}_B X^B{}_A$ is a 1-form on $O(M, g)$ which takes its values in the Lie algebra $gl(m, \mathbb{R})$ of $Gl(m, \mathbb{R})$ or in $o(m-p, p)$ of $O(m-p, p)$ and satisfies the structure equations

$$d\tilde{\omega} + \tfrac{1}{2}[\tilde{\omega}, \tilde{\omega}] = \tilde{\Omega} = \tilde{\text{H}}\text{or}\, d\tilde{\omega} \tag{1.18}$$

where $\tilde{\text{H}}\text{or}$ is understood in the sense of $\tilde{\omega}$ and $\tilde{\Omega} = \tilde{\Omega}^A{}_B X^B{}_A$ is a $gl(m, \mathbb{R})(o(m-p, p))$-valued 2-form of the curvature. We can write equation (1.18) using R^{2m}-valued forms and commutation relations of the Lie algebra $gl(m, \mathbb{R})(o(m, m-p))$,

$$\tilde{\Omega}^A{}_B = d\tilde{\omega}^A{}_B + \tilde{\omega}^A{}_C \wedge \tilde{\omega}^C{}_B \tag{1.19}$$

Taking any local section of $O(M, g)$, e, one can get the coefficients of the connection, curvature, basic forms, and torsion:

$$\begin{aligned} e^*\tilde{\omega}^A{}_B &= \omega^A{}_B \\ e^*\tilde{\Omega}^A{}_B &= \Omega^A{}_B \\ e^*\tilde{\theta}^A &= \theta^A \\ e^*\tilde{\Theta}^A &= \Theta^A \end{aligned} \tag{1.20}$$

The forms of the right-hand side of equations (1.20) are the forms defined in equations (1.11)–(1.14), etc. We call this formalism a linear (affine, metric, Riemannian, Einstein) connections on M.

In our theory it is necessary to consider at least four principal fiber bundles: a principal fiber bundle $\underline{P}$ over E with a structural group G (a gauge group), connection ω, and horizontally operator hor; a principal fiber bundle P' of frames over (E, g) with the connection $\tilde{\omega}^\alpha{}_\beta X^\beta{}_\alpha = \omega'$, a structural group $Gl(4, \mathbb{R})(O(1, 3))$, and an operator of horizontally $\overline{\text{hor}}$; a principal fiber bundle P'' of frames over $(\underline{P}, \gamma)$ (a metrized fiber bundle $\underline{P}$) with a structural group $Gl(4+n, \mathbb{R})(O(n+3, 1))$, a connection $\tilde{\omega}^A{}_B X^B{}_A = \tilde{\omega}$, and an operator of horizontality $\overline{\text{hor}}$; and a principal fiber bundle of frames P''' over G with a projection Π''', operator of horizontality $(\text{hor})'''$, a connection $\hat{\omega}$, and the structural group $Gl(n, \mathbb{R})$. Moreover, in order to simplify considerations, we use the formalism of linear connection coefficients on manifolds (E, g), $(\underline{P}, \gamma)$, and a principal fiber bundle formalism for $\underline{P}$, i.e., a principal fiber bundle over E with the structural group G a gauge group. I believe this is a way to make the formalism more natural and readable (see Fig. 1B).

2. THE NONSYMMETRIC TENSOR ON A LIE GROUP

Let G be a Lie group and define on G a tensor field $h = h_{ab} v^a \otimes v^b$ and a field of a 2-form $k = k_{ab} v^a \wedge v^b$, where

$$dv^a = -\tfrac{1}{2} C^a{}_{bc} v^b \wedge v^c \tag{2.1}$$

v^a is the usual left-invariant frame on G, and $C^a{}_{bc}$ are structure constants. Let X_a be generators of a Lie algebra $G - \mathfrak{g}$; X_a are left-invariant vector fields on G and they are dual to the forms v^a:

$$[X_a, X_b] = C^c{}_{ab} X_c \tag{2.2}$$

Using h and k, we construct a tensor field on G,

$$l_{ab} = h_{ab} + \mu k_{ab} \tag{2.3}$$

where μ is a real number. Recall that the left-invariant vector fields on G are infinitesimal transformations of a right action of G on G. The symbol $\mathrm{Ad}_G(g)$ means a matrix of the adjoint representation of the group G. For brevity we denote it $\mathrm{Ad}\, g$. We let R mean the right-action of the group G on G, and L the left-action $[R(g), L(g), g \in G]$.

We are looking for the following h and k:

$$R^*(g)h = h \tag{2.4}$$

$$R^*(g)k = k \tag{2.5}$$

or, in terms of the tensor l_{ab},

$$R^*(g)l = l \tag{2.6}$$

The condition (2.5) can be rewritten

$$(R^*(g))k_{g_1}(X_{g_1}, Y_{g_1}) = k_{g_1 g}(X_{g_1}g', Y_{g_1}g') = k_{g_1}(X_{g_1}, Y_{g_1}) \tag{2.5a}$$

where $g, g_1 \in G$.

Moreover, X, Y are left-invariant vector fields on G. Thus, $X_g = X_\varepsilon = X$, $Y_g = Y_\varepsilon = Y$, and

$$(R^*(g))k_{g_1}(X, Y) = k_{g_1 g}(Xg', Yg') = k_{g_1}(X, Y) \tag{2.5b}$$

where $\varepsilon \in G$ is a unit element of G.

In order to find h and k satisfying (2.4) and (2.5), we define a linear connection on G such that

$$\tilde{\omega}^a{}_b = -C^a{}_{bc} v^c \tag{2.7}$$

Let the covariant differentiation with respect to $\hat{\omega}^a{}_b$ be $\hat{\nabla}_c$ and an exterior covariant differentiation $\hat{D}$. It is easy to see that this connection is flat,

$$\hat{\Omega}^a{}_b = d\hat{\omega}^a{}_b + \hat{\omega}^a{}_c \wedge \hat{\omega}^c{}_b = 0 \tag{2.8}$$

with nonzero torsion

$$\hat{\Theta}^a = \hat{D}v^a = dv^a + \hat{\omega}^a{}_b \wedge v^c = \tfrac{1}{2} C^a{}_{bc} v^b \wedge v^c \tag{2.8a}$$

and with a tensor of torsion

$$\hat{Q}^a{}_{bc} = C^a{}_{bc} \tag{2.9}$$

This connection is also metric. It means that the Killing–Cartan tensor on the group G is absolutely parallel with respect to $\hat{\omega}^a{}_b$. A parallel transport according to this connection is a right-action of the group G on G.

One can easily find that (2.4)–(2.6) are equivalent to the condition

$$\hat{\nabla}_c l_{ab} = 0 \tag{2.10}$$

Thus in order to find h and k, we should solve equations (2.10) on the group G. Let us prove that the system (2.10) is self-consistent.

In order to do this, let us consider the commutator of the covariant derivatives

$$2\hat{\nabla}_{[r}\hat{\nabla}_{k]} l_{cd} = \hat{R}^b{}_{crk} l_{bd} + \hat{R}^b{}_{drk} l_{cb} + \hat{Q}^p{}_{rk} \hat{\nabla}_p l_{cd} \tag{2.11}$$

Moreover, $\hat{\omega}^a{}_b$ is flat and we get

$$2\hat{\nabla}_{[r}\hat{\nabla}_{k]} l_{cd} = \hat{Q}^p{}_{rk} \hat{\nabla}_p l_{cd} = C^p{}_{rk} \hat{\nabla}_p l_{cd}, \qquad \hat{\nabla}_p l_{cd} = 0 \tag{2.12}$$

which proves the consistency of (2.10)

We can get this result using the equivalent form of equation (2.10),

$$X_f l_{cd} + l_{nd} C^n_{cf} + l_{cn} C^n_{df} = 0 \tag{2.13}$$

It is easy to see that a bi-invariant tensor h on G satisfies (2.13) identically (for example, a Killing–Cartan tensor).

Thus, one gets for a tensor k_{ab}

$$\hat{\nabla}_c k_{ab} = X_c k_{ab} + k_{nb} C^n_{ac} + k_{an} C^n_{bc} = 0 \tag{2.14}$$

It is easy to see that if k_{ab} satisfies (2.14), $b \cdot k_{ab}$ satisfies this condition as well for $b = \text{const}$.

In the case of an Abelian group, k is bi-invariant on G.

The interesting case in our theory is a semisimple group G. In this case k_{ab} cannot be bi-invariant. The only bi-invariant 2-form on the semisimple Lie group G is a zero form. Moreover, equation (2.14) always has a solution on a semisimple group and k is right-invariant. Moreover, we suppose that the symmetric part of l is bi-invariant (left- and right-invariant) and k only right-invariant.

We can also define k in a special way,

$$k(A, B) = h([A, B], V), \qquad A = A^a X_a, \qquad B = B^a X_a \tag{2.15}$$

where

$$\hat{\nabla}_c V_d = 0 \tag{2.16}$$

$V = V_d \otimes v^d$ is a covector field on G (it is right-invariant) and h is a Killing–Cartan tensor on G.

In order to be more familiar with the notion of a tensor k, we find it for the group $SO(3)$.[62] In this case we have left-invariant vector fields

$$\begin{aligned} e_1 &= \cos\psi \frac{\partial}{\partial\theta} - \sin\psi\left(\cot\theta \frac{\partial}{\partial\psi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right) \\ e_2 &= \sin\psi \frac{\partial}{\partial\theta} + \cos\psi\left(\cot\theta \frac{\partial}{\partial\psi} - \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right) \\ e_3 &= \frac{\partial}{\partial\psi} \end{aligned} \tag{2.17}$$

such that

$$[e_a, e_b] = -\varepsilon_{abc} e_c; \qquad a, b, c = 1, 2, 3 \tag{2.18}$$

θ, ϕ, and ψ are Euler angles—the usual parametrization of $SO(3)$,

$$\begin{aligned} &0 \le \theta \le \pi \\ &0 \le \psi < 2\pi \\ &0 \le \phi < 2\pi \end{aligned} \tag{2.19}$$

and $\varepsilon_{123} = 1$, with ε_{abc} a Levi-Civita symbol.[62] In this case one can easily integrate (2.16); one finds

$$\begin{aligned} V_1(\theta, \phi, \psi) &= a(\cos\phi\cos\psi - \cos\theta\sin\phi\sin\psi) \\ &\quad + b\sin\psi\sin\theta - c(\sin\phi\cos\psi + \cos\theta\cos\phi\sin\psi) \\ V_2(\theta, \phi, \psi) &= a(\sin\psi\cos\phi + \cos\theta\cos\psi\sin\phi\,) \\ &\quad - b\cos\psi\sin\theta + c(\cos\theta\cos\phi\cos\psi - \sin\phi\sin\psi) \\ V_3(\theta, \phi, \psi) &= a\sin\phi\sin\theta + b\cos\theta + c\sin\phi\sin\theta, \qquad a, b, c = \text{const} \end{aligned} \tag{2.20}$$

In the simpler case $a=c=0$, $b\neq 0$, one gets

$$
\begin{aligned}
V_1&=b\sin\theta\sin\psi\\
V_2&=-b\sin\theta\cos\psi\\
V_3&=b\cos\theta,\qquad b=\text{const}
\end{aligned}
\tag{2.20a}
$$

For

$$k_{ab}=\varepsilon_{abc}V_c \tag{2.21}$$

we get

$$
k_{ab}=\begin{pmatrix}
0 & \begin{matrix}(a\sin\phi\sin\theta\\ +b\cos\theta\\ +c\sin\phi\sin\theta)\end{matrix} & \begin{matrix}-[a(\sin\psi\cos\phi\\ +\cos\theta\cos\psi\sin\phi)\\ -b\cos\psi\sin\theta\\ +c(\cos\theta\cos\phi\cos\psi\\ -\sin\phi\sin\psi)]\end{matrix}\\
\begin{matrix}-(a\sin\phi\sin\theta\\ +b\cos\theta\\ +c\sin\phi\sin\theta)\end{matrix} & 0 & \begin{matrix}[a(\cos\phi\cos\psi\\ -\cos\theta\sin\phi\sin\psi)\\ +b\sin\psi\sin\theta\\ -c(\sin\phi\cos\psi\\ +\cos\theta\cos\phi\sin\psi)]\end{matrix}\\
\begin{matrix}a(\sin\psi\cos\phi\\ +\cos\theta\cos\psi\sin\phi)\\ -b\cos\psi\sin\theta\\ +c(\cos\theta\cos\phi\cos\psi\\ -\sin\phi\sin\psi)\end{matrix} & \begin{matrix}-[a(\cos\phi\cos\psi\\ -\cos\theta\sin\phi\sin\psi)\\ +b\sin\psi\sin\theta\\ -c(\sin\phi\cos\psi\\ +\cos\theta\cos\phi\sin\psi)]\end{matrix} & 0
\end{pmatrix}
\tag{2.22}
$$

In the simpler case for $a=c=0$, $b\neq 0$, one gets

$$
k_{ab}=\begin{pmatrix}
0 & b\cos\theta & -b\sin\theta\cos\psi\\
-b\cos\theta & 0 & b\sin\theta\cos\psi\\
b\sin\theta\cos\psi & -b\sin\theta\sin\psi & 0
\end{pmatrix}
\tag{2.22a}
$$

Thus, if we choose for h a Killing–Cartan tensor on $SO(3)$ [this is a unique bi-invariant tensor on $SO(3)$ *modulo* constant factor],

$$h_{ab} = -2\delta_{ab} \tag{2.23}$$

we easily get

$$l_{ab} = \begin{pmatrix} -2 & \begin{matrix}\mu(\alpha \sin\phi \sin\theta \\ +\beta\cos\theta \\ +\gamma\sin\phi\sin\theta)\end{matrix} & \begin{matrix}-\mu[\alpha(\sin\psi\cos\phi \\ +\cos\theta\cos\psi\sin\phi) \\ -\beta\cos\psi\sin\theta \\ +\gamma(\cos\theta\cos\phi\cos\psi \\ -\sin\phi\sin\psi)]\end{matrix} \\ & -2 & \begin{matrix}\mu[\alpha(\cos\phi\cos\psi \\ -\cos\theta\sin\phi\sin\psi) \\ +\beta\sin\psi\sin\theta \\ -\gamma(\sin\phi\cos\psi \\ +\cos\theta\cos\phi\sin\psi)]\end{matrix} \\ \begin{matrix}\mu[\alpha(\sin\psi\cos\phi \\ +\cos\theta\cos\psi\sin\phi) \\ -\beta\cos\psi\sin\theta \\ +\gamma(\cos\theta\cos\phi\cos\psi \\ -\sin\phi\sin\psi)]\end{matrix} & \begin{matrix}-\mu[\alpha(\cos\phi\cos\psi \\ -\cos\theta\sin\phi\sin\psi) \\ +\beta\sin\psi\sin\theta \\ -\gamma(\sin\phi\cos\psi \\ +\cos\theta\cos\phi\sin\psi)]\end{matrix} & -2 \end{pmatrix} \tag{2.24}$$

where $\mu = \eta(a^2+b^2+c^2)^{1/2}$, $\eta^2 = 1$, $\alpha = a/\mu$, $\beta = b/\mu$, and $\gamma = c/\mu$. In the simpler case $a = c = 0$, $b \neq 0$, one gets (absorbing β by μ)

$$l_{ab} = \begin{pmatrix} -2 & \mu\cos\theta & \mu\sin\theta\cos\psi \\ -\mu\cos\theta & -2 & \mu\sin\theta\cos\psi \\ -\mu\sin\theta\cos\psi & -\mu\sin\theta\cos\psi & -2 \end{pmatrix} \tag{2.24a}$$

For an inverse tensor l^{ab} such that

$$l^{ab} l_{ac} = l^{ba} l_{ca} = \delta^b{}_c \tag{2.25}$$

we have

$$l^{ab} = \frac{\Delta^{ab}}{\Delta} \tag{2.26}$$

where $\Delta = \det(l_{ab}) = -2(4+\mu^2)$, Δ^{ab} is a cofactor matrix, and

$$\begin{aligned}
\Delta^{11} &= 4+\mu^2 \sin^2\theta \sin^2\psi \\
\Delta^{12} &= -(2\mu\cos\theta + \mu^2\sin^2\theta \sin\psi\cos\psi) \\
\Delta^{13} &= (\mu^2\cos\theta\sin\theta\sin\psi - 2\mu\sin\theta\cos\psi) \\
\Delta^{21} &= (2\mu\cos\theta - \mu^2\sin^2\theta\sin\psi\cos\psi) \\
\Delta^{22} &= (4+\mu^2\sin^2\theta\cos^2\psi) \\
\Delta^{23} &= -(2\mu\sin\theta\sin\psi + \mu^2\cos\theta\sin\theta\cos\psi) \\
\Delta^{31} &= (\mu^2\cos\theta\sin\theta\sin\psi + 2\mu\sin\theta\cos\psi) \\
\Delta^{32} &= (2\mu\sin\theta\sin\psi - \mu^2\cos\theta\sin\theta\cos\psi) \\
\Delta^{33} &= (4+\mu^2\cos^2\theta)
\end{aligned} \tag{2.27}$$

In the case of $SO(3)$, equation (2.22) is the most general tensor satisfying (2.5) except for a constant factor in front. Thus, this tensor is unique for $SO(3)$ *modulo* a constant factor.

In the case of any $SO(n)$ one can find k and l similarly using Euler angle parametrization and so for the classical groups $SU(n)$, $Sp(2n)$, G_2, F_4, E_6, E_7, E_8. In the case of solvable and nilpotent groups we can also try to find bi-invariant skew-symmetric tensors.

Finally, we suggest a general form of the tensor k_{ab} on a semi-simple group G, i.e., such that equation (2.4) is satisfied. The solutions of equations (2.10) and (2.14) are as follows:

$$l_{ab}(e^c) = l_{a'b'}(\varepsilon)(e^{Ad'C})^{a'}{}_a(e^{Ad'C})^{b'}{}_b$$

and

$$k_{ab}(e^C) = k_{a'b'}(\varepsilon)(e^{Ad'C})^{a'}{}_a(e^{Ad'C})^{b}{}_b$$

One writes

$$k_{ab}(g) = f_{a'b'}U^{a'}{}_a(g)U^{b'}{}_b(g), \qquad g\in G \tag{2.28}$$

where $U(g) = \mathrm{Ad}_G(g)$ is an adjoint representation of the group G. It is easy to see that for (2.28) we have

$$\hat{\nabla}_c k_{ab} = 0 \tag{2.29}$$

$$f_{ab} = -f_{ba} = \mathrm{const} \tag{2.30}$$

and it is defined in the representation space of the adjoint representation of the group G. In the case of the group $SO(3)$ one has

$$f_{ab} = \varepsilon_{abc} f_c \tag{2.31}$$

$$k_{ab} = \varepsilon_{abc} V_c \tag{2.31a}$$

and

$$V_a = f_{c'} U^{c'}{}_a(g) \tag{2.32}$$

If we choose $f_c = (0, 0, b)$, we get equation (2.20a). Moreover, this is always possible because an orthogonal $[SO(3)]$ transformation can transform any vector f into $(0, 0, \pm\|f\|)$, where $\|f\|$ is the length of f. The semisimple Lie group G can be considered a Riemannian manifold equipped with a bi-invariant tensor h (a Killing–Cartan tensor) and a connection induced by this tensor. This Riemannian manifold has a constant curvature. Such a manifold has a maximal group of isometries H of dimension $\frac{1}{2}n(n+1)$, $n = \dim G$.[59] (The isometry is here understood in the sense of the metric measured along geodetic lines in Riemannian geometry induced by a Killing–Cartan tensor.) This group is a Lie group. It is easy to see that for $G = SO(3)$ we have $H = SO(3) \otimes SO(3)$ and $\dim SO(3) \otimes SO(3) = 6$, $\dim SO(3) = 3$. The group $SO(3)$ leaves the Killing–Cartan tensor h_{ab} invariant

$$h_{a'b'} A^{a'}{}_a A^{b'}{}_b = h_{ab} \tag{2.33}$$

where $A \in SO(3)$.

Moreover, f_{ab} has exactly three arbitrary parameters and solutions of equation (2.14) have the same freedom in arbitrary constants. This suggests that the tensor (2.28) could be in some sense unique *modulo* an isometry on $SO(3)$ and a constant factor b. In this case the classification of k_{ab} tensors on an $SO(3)$ could be reduced to the classification of skew-symmetric tensors f_{ab} with respect to the action of the group $SO(3)$. In general, the situation is more complex, because $SO(n)$, $n = \dim G$, does not leave the commutator (Lie bracket) invariant.

Let us suppose that G is compact. In this case we should find all inequivalent f_{ab} tensors with respect to an orthogonal transformation $A \in SO(n)$. This means we should transform f_{ab} to a canonical form via an orthogonal matrix, i.e.,

$$(f_{ab}) = f \to f' = (f'_{ab}) = A^T f A = A^{-1} f A \tag{2.34}$$

For skew-symmetric matrices we have the following canonical forms, the so-called block-diagonal matrices.

For $n=2m$,

$$f=\begin{bmatrix} 0 & \xi^1 & & & \\ -\xi^1 & 0 & & \bigcirc & \\ & & \ddots & & \\ & \bigcirc & & 0 & \xi^m \\ & & & -\xi^m & 0 \end{bmatrix} \tag{2.35}$$

or, for $n=2m+1$,

$$f=\begin{bmatrix} 0 & \xi^1 & & & & \\ -\xi^1 & 0 & & & \bigcirc & \\ & & \ddots & & & \\ & & & 0 & \xi^m & 0 \\ & \bigcirc & & -\xi^m & 0 & 0 \\ & & & 0 & 0 & 0 \end{bmatrix} \tag{2.36}$$

where $\xi^1, \xi^2, \ldots, \xi^m$ are real numbers. In order to find them, we need to solve a secular equation for f,

$$\det(\mu I_n - f) = \mu^n + a_1(f)\mu^{n-2} + a_2(f)\mu^{n-2} + \cdots \tag{2.37}$$

$$I_n = (\delta^i_j)_{i,j=1,2,\ldots,n}$$

The coefficients $a_1, a_2, \ldots$ are invariant with respect to an action of the group $O(n)$ $[SO(n)]$ and they are functions of $\xi^1, \ldots, \xi^m$. Thus, in the case of a compact semisimple Lie group the skew-symmetric tensor k_{ab} on G is defined as follows:

$$k_{ab}(g) = b \cdot \tilde{f}_{a'b'} U^{a'}{}_a(g) U^{b'}{}_b(g) \tag{2.38}$$

where b is a constant real factor and $(\tilde{f}_{ab}) = \tilde{f}$ is given by

$$\tilde{f} = A^T \begin{bmatrix} 0 & 1 & & & & & \\ -1 & 0 & & & & \bigcirc & \\ & & 0 & \xi^1 & & & \\ & & -\xi^1 & 0 & & & \\ & & & & \ddots & & \\ & & \bigcirc & & & 0 & \xi^{m-1} \\ & & & & & -\xi^m - 1 & 0 \end{bmatrix} A \tag{2.39}$$

for $n=2m$, or

$$\tilde{f}=A^T\begin{bmatrix} 0 & 1 & & & & & & \\ -1 & 0 & & & & & & \\ & & 0 & \xi^1 & & \bigcirc & & \\ & & -\xi^1 & 0 & & & & \\ & & & & \ddots & & & \\ & & \bigcirc & & & 0 & \xi^{m-1} & \\ & & & & & -\xi^{m-1} & 0 & \\ & & & & & & & 0 \end{bmatrix}A \qquad (2.39a)$$

for $n=2m+1$.

Supposing that $h_{ab}=\operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where $n=2m$ or $n=2m+1$, one gets

$$l_{ab}(\varepsilon)=A^T\begin{bmatrix} \lambda_1 & \zeta^1 & & & & & \\ \zeta^1 & \lambda_2 & & & & & \\ & & \lambda_3 & \xi^2 & & \bigcirc & \\ & & -\xi^2 & \lambda_4 & & & \\ & & \bigcirc & & \ddots & & \\ & & & & & \lambda_{2m-1} & \xi^m \\ & & & & & -\zeta^m & \lambda_{2m} \end{bmatrix}A=A^T\tilde{l}(\varepsilon)A \qquad (2.39*)$$

for $n=2m$ or

$$l_{ab}(\varepsilon)=A^T\begin{bmatrix} \lambda_1 & \zeta^1 & & & & & & \\ -\zeta^1 & \lambda_2 & & & & & & \\ & & \lambda_3 & \xi^2 & & \bigcirc & & \\ & & -\xi^2 & \lambda_4 & & & & \\ & & \bigcirc & & \ddots & & & \\ & & & & & \lambda_{2m-1} & \xi^m & \\ & & & & & -\zeta^m & \lambda_{2m} & \\ & & & & & & & \lambda_{2m+1} \end{bmatrix}A=A^T\tilde{l}(\varepsilon)A$$

(2.39a∗)

for $n=2m+1$.

Moreover, if G is compact, we have $\lambda_i=\lambda$, $i=1, 2, \ldots, n$, and $\lambda<0$. This is because any bi-invariant symmetric tensor is proportional to the

Killing–Cartan tensor. In particular, the Tr tensor commonly used in Yang–Mills theory is proportional to h_{ab}. Thus, $h_{ab}=\lambda(\mathrm{Tr})_{ab}=\lambda\delta_{ab}$, $\lambda<0$. [For a particular normalization of generators, $\mathrm{Tr}(\{X_a, X_b\})=2\delta_{ab}$.] Let us remark that, in general, if $k_{ab}(\varepsilon)$ and h_{ab} commute (thus, for the moment, I do not suppose that G is compact), we have $l_{ab}(\varepsilon)=(A^{-1}\tilde{l}(\varepsilon)A)_{ab}$, where $A\,\varepsilon\, Gl(n, \mathbb{R})$ and $l_{ab}(g)=U^{a'}{}_a(g)U^{b'}{}_b(g)(A^{-1}\tilde{l}(\varepsilon)A)_{a'b'}$.

One can say, of course, that k_{ab} tensors are defined with more arbitrariness than are bi-invariant, symmetric tensors. This is because k is only right-invariant.

Let us notice that

$$f_{ab}=k_{ab}(\varepsilon) \tag{2.40}$$

(ε is the unit element of G) and

$$R_{g'}k_{ab}(g)=k_{ab}(gg')=k_{cd}(g)U^c{}_a)(g')U^d{}_b(g') \tag{2.41}$$

where $g, g'\in G$.

In the case of $G=SO(3)$, k_{ab} is unique up to an isometry of the Riemannian manifold with the bi-invariant tensor as a metric tensor and a constant factor. This suggests that the k_{ab} tensor given in the form (2.15)–(2.16) and (2.31)–(2.32) is an analogue of the Killing–Cartan tensor for k_{ab} (skew-symmetric). Moreover, the vector f can be transformed by an orthogonal $[O(n)]$ transformation into

$$\underbrace{(0, 0, \ldots, \pm\|f\|)}_{n \text{ times}} \tag{2.42}$$

Thus, one gets

$$k_{ab}(g)=b\cdot C^c_{ab}f^b_{c'}U^{c'}_c(g) \tag{2.43}$$

where b is a constant factor and

$$(f^0_{c'})=f^0=\underbrace{(0, 0, \ldots, 1)}_{n \text{ times}} \tag{2.44}$$

Thus, we can write k in a more compact form

$$k(A, B)(g)=b\cdot h([A, B], \mathrm{Ad}_g f^0) \tag{2.45}$$

where $A=A^aX_a$, $B=B^aX_a$.

Using the bi-invariancy of the Killing–Cartan tensor, one can write

$$k(A, B)(g)=b\cdot h(\mathrm{Ad}_{g^{-1}}[A, B], f^0) \tag{2.45a}$$

Moreover, if there is $\tilde{g}\in G$ such that $\tilde{g}^2=g$, we get

$$k(A, B)(\tilde{g}^2)=b\cdot h(\mathrm{Ad}_{\tilde{g}^{-1}}[A, B], \mathrm{Ad}_{\tilde{g}}f^0) \tag{2.46}$$

We find an interpretation of the factor b for K given by formulas (2.45)–(2.46).

We get

$$k_{ab}k^{ab}=h^{aa'}h^{bb'}k_{ab}k_{a'b'}=b^2\|\mathrm{Ad}_g f^0\|^2=b^2 \tag{2.47}$$

Thus, we have

$$b=\pm(k_{ab}k^{ab})^{1/2} \tag{2.48}$$

Finally, let us notice that we can repeat the considerations changing right (left)-invariant to left (right)-invariant in all places. In this case we can consider a left-invariant 2-form k and a left-invariant nonsymmetric tensor on a Lie group G.

3. THE NONSYMMETRIC METRIZATION OF THE BUNDLE $\underline{P}$

Let us introduce the principal fiber bundle $\underline{P}$ over the space-time E with the structural group G and with the projection π. Let us suppose that $(E, \mathbf{g})$ is a manifold with a nonsymmetric metric tensor of the signature $(-,-,-,+)$,

$$\mathbf{g}_{\mu\nu}=\mathbf{g}_{(\mu\nu)}+\mathbf{g}_{[\mu\nu]} \tag{3.1}$$

Let us introduce a natural frame on $\underline{P}$,

$$\theta^A=(\pi^*(\bar{\theta}^\alpha),\ \theta^a=\lambda\omega^a), \qquad \lambda=\mathrm{const} \tag{3.2}$$

It is convenient to introduce the following notations. Capital Latin indices A, B, C run over $1, 2, 3, \ldots, n+4$, $n=\dim G$. Lower case Greek indices take the values $\alpha, \beta, \gamma, \delta=1, 2, 3, 4$ and lower case Latin cases $a, b, c, d=5, 6, \ldots, n+4$. The symbol bar over θ^α and other quantities indicates that these quantities are defined on E.

It is easy to see that the existence of the nonsymmetric metric on E is equivalent to introducing two independent geometrical quantities on E,

$$\bar{\mathbf{g}}=\mathbf{g}_{\alpha\beta}\bar{\theta}^\alpha\otimes\bar{\theta}^\beta=\mathbf{g}_{(\alpha\beta)}\bar{\theta}^\alpha\otimes\bar{\theta}^\beta \tag{3.3}$$

$$\underline{\mathbf{g}}=g_{\alpha\beta}\,\bar{\theta}^\alpha\wedge\bar{\theta}^\beta=\mathbf{g}_{[\alpha\beta]}\bar{\boldsymbol{\theta}}^\alpha\wedge\bar{\boldsymbol{\theta}}^\beta \tag{3.4}$$

i.e., the symmetric metric tensor $\bar{\mathbf{g}}$ on E and 2-form $\underline{\mathbf{g}}$. On the group G we can introduce a bi-invariant symmetric tensor called the Killing–Cartan tensor,

$$h(A, B)=\mathrm{Tr}(\mathrm{Ad}'_A\circ\mathrm{Ad}'_B) \tag{3.5}$$

where $\mathrm{Ad}'_A(C)=[A, C]$ (it is tangent to Ad, i.e., it is an "infinitesimal" Ad transformation). It is easy to see that

$$h(A, B)=h_{ab}A^a\cdot B^b \tag{3.6}$$

where

$$h_{ab}=C^c{}_{ad}C^d{}_{bc}, \qquad h_{ab}=h_{ba}, \qquad A=A^aX_a, \qquad B=B^aX_a$$

This tensor is distinguished by the group structure, but there are of course other bi-invariant tensors on G. Normally, it is supposed that G is semi-simple. This means that $\det(h_{ab})\neq 0$. In this construction we use $l_{(ab)}=h_{ab}$ (the bi-invariant tensor on G) in order to get a proper limit (i.e., the non-Abelian Kaluza–Klein theory) for $\mu=0$.

For a natural 2-form k on G, or a natural skew-symmetric right-invariant tensor, we choose k described in Section 2; k is zero for $\mathbf{U}(1)$. Let us turn to the nonsymmetric natural metrization of $\underline{P}$. Let us suppose that

$$\bar{\gamma}(X, Y)=\bar{\mathbf{g}}(\pi'X, \pi'Y)+\lambda^2\rho^2h(\omega(X), \omega(Y)) \tag{3.7}$$

$$\underline{\gamma}(X, Y)=\underline{\mathbf{g}}(\pi'X, \pi'Y)+\mu\lambda^2\rho^2k(\omega(X), \omega(Y)) \tag{3.8}$$

$\mu=\text{const}$ and its dimensionless, X, $Y\in\text{Tan}(\underline{P})$, and $\rho=\rho(x)$ is a scalar field defined on E. The formula (3.7) was introduced by A. Trautman (in the case with $\rho=1$) for the symmetric natural metrization of $\underline{P}$ and it was used to construct the Kaluza–Klein theory for $\mathbf{U}(1)$ and non-Abelian generalizations of this theory. It is easy to see that

$$\bar{\gamma}=\pi^*\bar{\mathbf{g}}\otimes\rho^2h_{ab}\theta^a\otimes\theta^b \tag{3.9}$$

$$\underline{\gamma}=\pi^*\underline{\mathbf{g}}+\mu\rho^2k_{ab}\theta^a\wedge\theta^b \tag{3.10}$$

or

$$\gamma_{(AB)}=\left(\begin{array}{c|c}\mathbf{g}_{[\alpha\beta]} & 0\\ \hline 0 & \rho^2h_{ab}\end{array}\right) \tag{3.11}$$

$$\gamma_{[AB]}=\left(\begin{array}{c|c}\mathbf{g}_{\alpha\beta} & 0\\ \hline 0 & \mu\rho^2k_{ab}\end{array}\right) \tag{3.12}$$

For

$$\gamma_{AB}=\gamma_{(AB)}+\gamma_{[AB]}$$

one easily gets

$$\gamma_{AB}=\left(\begin{array}{c|c}\mathbf{g}_{\alpha\beta} & 0\\ \hline 0 & \rho^2l_{ab}\end{array}\right) \tag{3.13}$$

where $l_{ab}=h_{ab}+\mu k_{ab}$. The tensor γ_{AB} has this simple form in the natural frame on $\underline{P}$, U^A. This frame is unholonomical, because

$$d\theta^a=\frac{\lambda}{2}(H^a{}_{\mu\nu}\theta^\mu\wedge\theta^\nu-\frac{1}{\lambda^2}C^a{}_{bc}\theta^b\wedge\theta^c)\neq 0 \tag{3.14}$$

γ is invariant with respect to the right-action of the group on $\underline{P}$. In the case with $k_{ab}=0$ we have

$$\gamma_{AB}=\left(\begin{array}{c|c}\mathbf{g}_{\alpha\beta} & 0\\ \hline 0 & \rho^2 h_{ab}\end{array}\right) \tag{3.15}$$

For the electromagnetic case $[G=\mathbf{U}(1)]$ one easily finds

$$\gamma_{AB}=\left(\begin{array}{c|c}\mathbf{g}_{\alpha\beta} & 0\\ \hline 0 & -\rho^2\end{array}\right) \tag{3.16}$$

Now let us take a section $e: E\to\underline{P}$ and attach to it a frame υ^a, $a=5,6,\ldots,n+4$, selecting $X^\mu=$const on a fiber in such a way that e is given by the condition $e^*\upsilon^a=0$ and the fundamental fields ζ_α such that $\upsilon^a(\zeta_\beta)=\delta^a_b$ satisfy $[\zeta_b,\zeta_a]=(1/\lambda)C^c{}_{ab}\zeta_c$.

Thus, we have

$$\omega=\frac{1}{\lambda}\upsilon^a X_a+\pi^*(A^a{}_\mu\bar{\theta}^\mu)X_a$$

where

$$e^*\omega=A=A^a{}_\mu\bar{\theta}^\mu X_a$$

In this frame the tensor takes the form

$$\gamma_{AB}=\left(\begin{array}{c|c}\mathbf{g}_{\alpha\beta}+\lambda^2\rho^2 l_{ab}A^a_\alpha A^b_\beta & \lambda\rho^2 l_{cb}A^c_\alpha\\ \hline \lambda\rho^2 l_{ac}A^c{}_\beta & \rho^2 l_{ab}\end{array}\right) \tag{3.17}$$

where

$$l_{ab}=h_{ab}+\mu k_{ab}$$

This frame is also unholonomic. One easily finds

$$d\upsilon^a=\frac{-1}{2\lambda}C^a{}_{bc}\upsilon^b\wedge\upsilon^c \tag{3.18}$$

The nonsymmetric theory of gravitation uses the nonsymmetric metric $\mathbf{g}_{\mu\nu}$ such that

$$\mathbf{g}_{\mu\nu}\mathbf{g}^{\beta\nu}=\mathbf{g}_{\nu\mu}\mathbf{g}^{\nu\beta}=\delta^\beta{}_\mu \tag{3.19}$$

where the order of indices is important. If G is semisimple and $k_{ab}=0$,

$$l_{ab}=h_{ab}, \qquad \det(h_{ab})\neq 0$$

and

$$h_{ab}h^{bc}=\delta_a^c \tag{3.20}$$

Thus, one easily finds in this case

$$\gamma_{AC}\gamma^{BC}=\gamma_{CA}\gamma^{CB}=\delta_A^B \tag{3.21}$$

where the order of indices is important. We have the same for the electromagnetic case $[G=\mathrm{U}(1)]$. In general, if $\det(l_{ab})\neq 0$, then

$$l_{ab}l^{ac}=l_{ba}l^{ca}=\delta_b^c \tag{3.22}$$

where the order of indices is important. From (3.22) we have (3.21) for the general nonsymmetric metric γ.

It is easy to see that

$$\begin{aligned}\Phi'(\mathrm{g})\bar{\gamma}&=\bar{\gamma}\\ \Phi'(\mathrm{g})\underline{\gamma}&=\underline{\gamma}\end{aligned} \tag{3.23}$$

and γ_{AB} is an invariant tensor with respect to the right-action of the group G on $\underline{P}$.

In the case of any Abelian group the condition (3.23) is stronger and we get that γ_{AB} is bi-invariant. Thus, in the case of $G=\mathrm{U}(1)$ (electromagnetic case)

$$\xi_5\bar{\gamma}=0=\xi_5\underline{\gamma} \tag{3.24}$$

where ξ_A is a dual base

$$\theta^A(\xi_B)=\delta_B^A \tag{3.25}$$

$A, B=1, 2, 3, 4, 5,$

$$\xi_A=(\xi_\alpha, \xi_5) \tag{3.26}$$

Let us come back to the connection $\hat{\omega}^a{}_b$ defined on the group G. For a typical fiber that is diffeomorphic to G, we can define $\hat{\omega}^a{}_b$ on every fiber $F_x\simeq G$, $x\in E$. Due to a local trivialization of the bundle $\underline{P}$, we can define $\hat{\omega}^a{}_b$ on every set $U\times G$, where $U\subset E$ and is open. Thus, we get a linear connection on P such that

$$\hat{\omega}^A{}_B=\left(\begin{array}{c|c}0 & 0\\ \hline 0 & -(1/\lambda)C^a{}_{bc}\theta^c\end{array}\right) \tag{3.27}$$

defined in a frame $\theta^A=(\pi^*(\bar{\theta}^\alpha), \theta^a)$, where $\bar{\theta}^\alpha$ is a frame on E and θ^a is a horizontal lift base.

This connection can be examined in a systematic way. Let us introduce a metric on P in the following way:

$$p = \pi^* \eta \oplus h_{ab} \theta^a \otimes \theta^b \tag{3.28}$$

where $\eta = \eta_{\mu\nu} \bar{\theta}^\mu \otimes \bar{\theta}^\nu$ is a Minkowski tensor and h_{ab} is a Killing–Cartan tensor on G. We get

$$p_{AB} = \left(\begin{array}{c|c} \eta_{ab} & 0 \\ \hline 0 & h_{ab} \end{array}\right) \quad \text{and} \quad p^{AB} = \left(\begin{array}{c|c} \eta^{\alpha\beta} & 0 \\ \hline 0 & h^{ab} \end{array}\right) \tag{3.29}$$

The connection $\hat{\omega}^A{}_B$ can be defined as

$$\hat{\omega}^A{}_B = \left(\begin{array}{c|c} \pi^* \hat{\bar{\omega}}^\alpha{}_\beta & 0 \\ \hline 0 & \phi_x^* \hat{\omega}^a{}_b \end{array}\right) \tag{3.27a}$$

where $\hat{\bar{\omega}}^\alpha{}_\beta$ is a trivial connection on the Minkowski space, $\hat{\omega}^a{}_b$ is the connection defined in Section 2.2, and ϕ_x is a diffeomorphism $\phi_x\colon F_x \to G$, $x \in U$.

It is easy to check that

$$\hat{D} p_{AB} = 0 = \hat{D} p^{AB} \tag{3.30}$$

where $\hat{D}$ is an exterior covariant differential with respect to $\hat{\omega}^A{}_B$. One can easily calculate the torsion for $\hat{\omega}^A{}_B$,

$$\hat{Q}^a{}_{\mu\nu} = \lambda H^a{}_{\mu\nu} \tag{3.31a}$$

$$\hat{Q}^a{}_{bc} = \frac{1}{\lambda} C^a{}_{bc} \tag{3.31b}$$

and the curvature tensor

$$\hat{R}^a{}_{b\mu\nu} = \lambda X_b H^a{}_{\mu\nu} \tag{3.32}$$

(the remaining torsion and curvature components are zero).

The connection $\hat{\omega}^A{}_B$ is neither flat nor torsionless. Moreover, it is still metric as a connection $\hat{\omega}^a{}_b$ from Section 2.2.

The covariant differentiation with respect to this connection is connected to the right-action of the group G on P. Thus, the condition of the right-invariance of the p-form $\Xi^{A_1 \ldots A_l}{}_{B_1 \ldots B_m}$ on P is equivalent to

$$\hat{\nabla}_a \Xi^{A_1 \ldots A_l}{}_{B_1 \ldots B_m} = 0 \tag{3.33}$$

where $\hat{\nabla}_k$ is a covariant derivative with respect to $\hat{\omega}^A{}_B$ in vertical directions on P. This means right-invariance of Ξ. This can be written

$$\hat{\nabla}_{\mathrm{ver}(x)} \Xi = 0 \tag{3.33a}$$

ver is understood in the sense of ω. We have

$$\Phi'(g)\Xi=\Xi \tag{3.34}$$

where $g\in G$ and

$$\Xi=(\Xi^{A_1\dots A_l B_1\dots B_m})=(p^{B_1B_1'}p^{B_2B_2'}\dots p^{B_mB_m'}\Xi^{A_1\dots A_l}{}_{B_1'\dots B_m'})$$

For a connection ω on a bundle P, of curvature Ω, one gets

$$\hat{\nabla}_k\omega=\hat{\nabla}_k\Omega=0 \tag{3.34*}$$

Thus, we can rewrite equation (3.23)

$$\hat{\nabla}_a\underline{\gamma}=\hat{\nabla}_a\bar{\gamma}=0 \tag{3.35}$$

This means that

$$\hat{\nabla}_a\gamma_{AB}=0 \tag{3.36}$$

or

$$\hat{\nabla}_{\mathrm{ver}(x)}\gamma=0 \tag{3.36a}$$

For every linear connection $\omega^A{}_B$ defined on P compatible in some sense with γ_{AB} we get

$$\Phi^*(g)\omega_{AB}=\mathrm{Ad}_g\omega_{AB} \tag{3.37}$$

which means that ω_{AB} is right-invariant with respect to the right-action of the group G on P. We say the same for the 2-form of torsion and the 2-form of curvature derived for $\omega^A{}_B$, i.e.,

$$\hat{\nabla}_a\Omega^A{}_B=\hat{\nabla}_a\Theta^A=0 \tag{3.38}$$

The curvature scalar is invariant with respect to the right-action of the group G on P,

$$0=\hat{\nabla}_aR=X_aR=\zeta_aR \tag{3.39}$$

The condition (3.37) is the same as in the classical Kaluza–Klein (Jordan–Thiry) theory in the non-Abelian case. A parallel transport with respect to the connection $\hat{\omega}^A{}_B$ means simply a right-action of the group G on P.

Our subject of investigation consists in looking for a generalization of the geometry from Einstein's unified field theory (the so-called Einstein–Kaufman theory[4,5,61]) defined on P, i.e., for a connection $\omega^A{}_B$ such that

$$D\gamma_{AB}=\gamma_{AD}Q^D{}_{BE}\theta^E \tag{3.40}$$

where D is the exterior, covariant differential with respect to the connection $\omega^A{}_B$ and $Q^D{}_{BE}$ is the tensor for $\omega^A{}_B$. We suppose that this connection is right-invariant with respect to the right-action of the group G.

We can write equations (3.37)–(3.39) for the torsion, curvature, and scalar of curvature for $\omega^A{}_B$. In this way we consider the Einstein–Kaufman G-structure on the bundle of linear frames over the manifold $\underline{P}$ (i.e., a right G-structure).

We can repeat all the considerations changing right (left)-invariant into left (right)-invariant in all places.

In this section we define $\omega^A{}_B$ as a collection of 1-forms defined on the manifold $\underline{P}$ (a gauge bundle manifold) and we choose for $\omega^A{}_B$ a lift horizontal frame (connected to the connection ω on the gauge bundle).

The collection of 1-forms $\omega^A{}_B$ becomes a linear connection on $\underline{P}$ iff it satisfies the following transformation properties:

$$\omega'^{A'}{}_{B'} = \Sigma^{-1A'}{}_A(p)\omega^A{}_B\Sigma^{-1B}{}_{B'}(p) - \Sigma^{-1A'}{}_A(p)d\Sigma^A{}_{B'}(p) \tag{3.41}$$

where

$$\Sigma(p) \in GL(n+4, \mathbb{R}), \qquad p \in U_p \subset \underline{P}$$

and

$$\theta^c = \Sigma^c{}_{c'}(p)\theta'^{c'} \tag{3.42}$$

is a simultaneous transformation propery of a frame. Having $\omega^A{}_B$ with transformation properties (3.41)–(3.42), we can lift it on a principal fiber bundle of frames over $\underline{P}$ with the structural group $GL(n+4, \mathbb{R})$, getting a 1-form of connection $\tilde{\omega}$,

$$\tilde{\omega}_z = \mathrm{Ad}_{GL(n+4,R)}(g_p^{-1})[\Pi^*(\omega^A{}_B X^B{}_A) - g_p^{-1}\, dg_p] \tag{3.43}$$

where Π is a projection defined on this principal fiber bundle of frames and

$$g_p\colon \quad z \in \Pi^{-1}(U_p) \to g_p(z) = (\mathrm{pr}_{GL(n+4,R)}\Psi_p(z))^{-1}$$
$$\in \mathrm{Gl}(n+4, \mathbb{R}), \qquad p \in U_p \subset \underline{P}$$

pr means a projection on $Gl(n+4, \mathbb{R})$ in a local trivialization of the bundle P'', Ψ is an action of $GL(n+4, \mathbb{R})$ on a principal fiber bundle of frames over $\underline{P}$, $\Psi \to Gl(n+4, \mathbb{R}) \times P'' \to P''$, and Ψ_p is defined for $Gl(n+4, \mathbb{R}) \times U_p$. In this way we have an action of $GL(n+4, \mathbb{R})$ on the bundle and for $\tilde{\omega}$,

$$\Psi^*(g)\tilde{\omega} = \mathrm{Ad}_{GL(n+4,R)}(g^{-1})\tilde{\omega} \tag{3.44}$$

$X^A{}_B$ are generators of the Lie algebra $gl(n+4, \mathbb{R})$ of $GL(n+4, \mathbb{R})$ and $g \in Gl(n+4, \mathbb{R})$. For a soldering form $\tilde{\theta}^A$ one gets

$$\tilde{\theta}^A = g_p \Pi^*(\theta^A) \tag{3.45}$$

Taking any two sections of the principal fiber bundle of P'' frames E and F such that

$$E^*\tilde{\omega} = \omega'^{A'}{}_{B'}X^{B'}{}_{A'}$$
$$F^*\tilde{\omega} = \omega^{A}{}_{B}X^{B}{}_{A} \tag{3.46}$$

$$E^*\tilde{\Theta}^A = \theta'^A$$
$$F^*\tilde{\theta}^A = \theta^A \tag{3.47}$$

one gets the transformation properties (3.41) and (3.42). In such a way that

$$E(p) = F(p)\Sigma(p) \tag{3.48}$$

equation (3.40) can be rewritten in a more compact form

$$\nabla\gamma = S \tag{3.49}$$

where

$$S(X, Y, Z) = [\mathrm{Tr}(\gamma \otimes Q)](X, Y, Z) = \sum_A \gamma(X, e_A)\theta^A(Q(Y, Z))$$
$$Q(Y, Z) = -Q(Z, Y)$$

is the torsion of the connection $\tilde{\omega}$, X, Y, Z are contravariant vector fields; and θ^A and e_B, $\theta^A(e_B) = \delta^A{}_B$, are dual bases.

Or, in a different form,

$$\nabla_z \gamma(X, Y) = S(X, Y, Z) \tag{3.50}$$

∇ is a covariant derivative with respect to the connection $\tilde{\omega}$ on the fiber bundle of frames.

Moreover, now we consider γ, Q, X, Y, Z, etc., as geometrical objects living on appropriate associated fiber bundles to the fiber bundle of frames. The condition (3.50) gives us the Einstein–Kaufman connection $\tilde{\omega}$ on the principal fiber bundle of frames over P. For $\tilde{\omega}$ right-invariant with respect to the action of a group G on this bundle of frames (lifted to this bundle from $\underline{P}$), the condition (3.50) is also right-invariant.

4. FORMULATION OF THE NONSYMMETRIC JORDAN-THIRY THEORY

Let $\underline{P}$ be the principal fiber bundle with the structural group $G = \mathrm{U}(1)$ over space-time E with a projection π and let us define on this bundle a connection α. We call this bundle an electromagnetic bundle and α an electromagnetic connection. For the electromagnetic bundle $\underline{P}$ we can specify all quantities introduced in Sections 1–3. We have

$$\Omega = d\alpha = \tfrac{1}{2}\pi^*(F_{\mu\nu}\bar{\theta}^\mu \wedge \bar{\theta}^\nu)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \qquad e^*\alpha = A_\mu \bar{\theta}^\mu$$

and e is a local section of $\underline{P}$. Here A_μ is the four-potential of the electromagnetic field and $F_{\mu\nu}$ is its strength. The Bianchi identity is

$$d\Omega = 0 \tag{4.*}$$

and due to this, the four-potential exists. It is of course the first Maxwell equation.

On space-time E we define a nonsymmetric metric tensor $\mathbf{g}_{\alpha\beta}$ such that

$$\begin{aligned} \mathbf{g}_{\alpha\beta} &= \mathbf{g}_{(\alpha\beta)} + \mathbf{g}_{[\alpha\beta]} \\ \mathbf{g}_{\alpha\beta}\mathbf{g}^{\gamma\beta} &= \mathbf{g}_{\beta\alpha}\mathbf{g}^{\beta\gamma} = \delta^\gamma_\alpha \end{aligned} \tag{4.1}$$

where the order of indices is important. We define also on E two connections $\bar{\omega}^\alpha{}_\beta$ and $\bar{W}^\alpha{}_\beta$,

$$\bar{\omega}^\alpha{}_\beta = \bar{\Gamma}^\alpha{}_{\beta\gamma}\bar{\theta}^\gamma \tag{4.2}$$

and

$$\bar{W}^\alpha{}_\beta = \bar{W}^\alpha{}_{\beta\gamma}\bar{\theta}^\gamma = \bar{\omega}^\alpha{}_\beta - \tfrac{2}{3}\delta^\alpha{}_\beta \bar{W} \tag{4.3}$$

where

$$\bar{W} = \bar{W}_\gamma \bar{\theta}^\gamma = \tfrac{1}{2}(\bar{W}^\sigma{}_{\gamma\sigma} - \bar{W}^\sigma{}_{\sigma\gamma})\bar{\theta}^\gamma$$

For the connection $\bar{\omega}^\alpha{}_\beta$ we suppose the following conditions:

$$\begin{aligned} \bar{D}\mathbf{g}_{\alpha+\beta-} = \bar{D}\mathbf{g}_{\alpha\beta} - \mathbf{g}_{\alpha\delta} \cdot \bar{Q}^\delta{}_{\beta\gamma}(\bar{\Gamma})\theta^\gamma &= 0 \\ \bar{Q}^\alpha{}_{\beta\alpha}(\bar{\Gamma}) &= 0 \end{aligned} \tag{4.4}$$

where $\bar{D}$ is the exterior covariant derivative with respect to $\bar{\omega}^\alpha$ and $\bar{Q}^\alpha{}_{\beta\gamma}(\bar{\Gamma})$ is the torsion of $\bar{\omega}^\alpha{}_\beta$.

Thus, we have on the space-time E all the quantities from NGT.[34,63,64] Now let us turn to the natural nonsymmetric metrization of the bundle $\underline{P}$. According to Section 1, we have

$$\begin{aligned} \bar{\gamma} &= \pi^*\bar{\mathbf{g}} - \rho^2\theta^5 \otimes \theta^5 = \pi^*(\mathbf{g}_{(\alpha\beta)}\bar{\theta}^\alpha \otimes \bar{\theta}^\beta) - \rho^2\theta^5 \otimes \theta^5 \\ \underline{\gamma} &= \pi^*\underline{\mathbf{g}} = \pi^*(\mathbf{g}_{[\alpha\beta]}\bar{\theta}^\alpha \wedge \bar{\theta}^\beta) \end{aligned} \tag{4.5}$$

where $\theta^5 = \lambda\alpha$. From the classical Kaluza–Klein theory we know that[16] $\lambda = 2$ (we work with an appropriate system of units), and we have

$$\gamma_{AB} = \left(\begin{array}{c|c} g_{\alpha\beta} & 0 \\ \hline 0 & -\rho^2 \end{array}\right) \tag{4.6}$$

The tensor γ_{AB} has this shape in a lift horizontal base, which is of course nonholonomic. We can find it in a holonomic system of coordinates. Let us take a section $e: E \to P$ and attach to it a coordinate x^5, selecting $x^\mu = \text{const}$ on the fiber in such a way that e is given by the condition $x^5 = 0$ and $\zeta_5 = \partial/\partial x^5$. Then we have $e^* \, dx^5 = 0$ and

$$\alpha = \frac{1}{\lambda} dx^5 + \pi^*(A_\mu \bar{\theta}^\mu), \qquad \text{where} \qquad A = A_\mu \bar{\theta}^\mu = e^* \alpha.$$

In this coordinate system the tensor γ takes the form

$$\gamma_{AB} = \left(\begin{array}{c|c} g_{\alpha\beta} - \lambda^2 \rho^2 A_\alpha A_\beta & -\lambda \rho^2 A_\alpha \\ \hline -\lambda \rho^2 A_\beta & -\rho^2 \end{array}\right) \tag{4.6a}$$

In order to have the correct dimension of a four-potential we should rather write $e^*\alpha = (q/\hbar c)A = \mu A$, where q is an elementary charge and $\hbar$ is Planck's constant. The same is true for the curvature of connection on the electromagnetic bundle $\Omega = \lambda\mu\pi^*(F)$, $F = \frac{1}{2} F_{\mu\nu} \bar{\theta}^\mu \wedge \bar{\theta}^\nu$. Moreover, it can be absorbed by a constant λ.

Now we define on P a connection $\omega^A{}_B$ bi-invariant with respect to the action of the group U(1) on P, such that

$$\begin{aligned} D\gamma_{A+B-} &= D\gamma_{AB} - \gamma_{AD} Q^D{}_{BC}(\Gamma)\theta^C = 0 \\ \omega^A{}_B &= \Gamma^A{}_{BC} \theta^C \end{aligned} \tag{4.7}$$

D is the exterior covariant derivative with respect to the connection $\omega^A{}_B$, and $Q^D{}_{BC}(\Gamma)$ is the tensor of torsion for the connection $\omega^A{}_B$.

After some calculations one finds

$$\omega^A{}_B = \left(\begin{array}{c|c} \pi^*(\bar{\omega}^\alpha{}_\beta) + \rho^2 g^{\delta\alpha} H_{\delta\beta} \theta^5 & H_{\beta\gamma}\theta^\gamma + (1/\rho) g_{\beta\delta} \tilde{g}^{(\gamma\delta)} \rho_{,\gamma} \theta^5 \\ \hline \rho^2 g^{\alpha\beta}(H_{\gamma\beta} - 2F_{\gamma\beta})\theta^\gamma + \rho \tilde{g}^{(\gamma\alpha)} \rho_{,\gamma} \theta^5 & (1/\rho) g_{\delta\gamma} \tilde{g}^{(\alpha\delta)} \rho_{,\alpha} \theta^\gamma \end{array}\right) \tag{4.8}$$

where $\tilde{g}^{(\alpha\delta)}$ is the inverse tensor for $g_{(\alpha\beta)}$, i.e., $\tilde{g}^{(\alpha\delta)} \cdot g_{(\alpha\beta)} = \delta^{\delta\beta}$, and $H_{\gamma\beta}$ is a tensor defined on E such that

$$g_{\delta\beta} g^{\gamma\delta} H_{\gamma\alpha} + g_{\alpha\delta} g^{\delta\gamma} H_{\beta\gamma} = 2 g_{\alpha\delta} g^{\delta\gamma} F_{\beta\gamma} \tag{4.9}$$

We define on P a second connection such that

$$W^A{}_B = \omega^A{}_B - \tfrac{4}{9} \delta^A_B \bar{W} \tag{4.10}$$

where $\bar{W}$ is defined in (4.3). It is easy to see that $\bar{W}$ is a horizontal 1-form

$$\bar{W} = \text{hor } \bar{W} \tag{4.11}$$

(horizontality is understood in the sense of the connection α defined on the bundle $\underline{P}$). Thus, we have on $\underline{P}$ all five-dimensional analogues of the quantities from Moffat's theory of gravitation i.e., $W^A{}_B$, $\omega^A{}_B$, and γ_{AB}.[63–65]

They are also analogues to the quantities from Einstein's unified field theory. γ_{AB}, $\omega^A{}_{.B}$, and $W^A{}_{.B}$ are $E(1)$-invariant. Thus, we get the Einstein–Kaufman U(1) structure.

5. GEODETIC EQUATION

Let us write an equation for geodesics $\Gamma \subset P$ with respect to the connection $\omega^A{}_B$ on $\underline{P}$.

$$\begin{gathered} \nabla_U u = 0 \\ u^B \nabla_B u^A = 0 \end{gathered} \tag{5.1}$$

where u $[U^A(t)]$ is a tangent vector to the geodesic line and ∇ means a covariant derivative with respect to the connection $\omega^A{}_B$. Using (4.8), one easily finds

$$\frac{\bar{D}u^\alpha}{dt} + u^5\rho^2(g^{\alpha\mu}(H_{\beta\mu} - 2F_{\beta\mu}) + g^{\mu\alpha}H_{\mu\beta})u^\beta + (u^5)^2 \cdot \rho \cdot \tilde{g}^{(\beta\alpha)}\rho_{,\beta} = 0 \tag{5.2}$$

$$\frac{du^5}{dt} + \frac{2u^5}{\rho}\frac{d\rho}{dt} + \rho^2 H_{\gamma\beta}u^\gamma u^\beta = 0 \tag{5.3}$$

where D/dt means covariant derivative with respect to $\bar{\omega}^\alpha{}_\beta$ along a curve to which $u(t)$ is tangent and

$$\frac{d\rho}{dt} = \rho_{,\beta}u^\beta \tag{5.4}$$

One easily transforms (5.3) into

$$\frac{d}{dt}(2u^5\rho^2) + \tfrac{1}{2}H_{\gamma\beta}u^\gamma u^\beta = 0 \tag{5.5}$$

It is easy to see that (5.5) has a first integral

$$2u^5\rho^2 = \text{const} \qquad \text{iff} \quad H_{\gamma\beta} = -H_{\beta\gamma} \tag{5.6}$$

In the Kaluza–Klein theory or in the Jordan–Thiry theory, $2u^5\rho^2$ has an interpretation as q/m_0 for a test particle, where q is the charge and m_0 is the rest mass of the test particle. This first integral means that q/m_0 does not change during the movement of the test particle. Finally we get

$$\frac{\tilde{D}u^\alpha}{dt}+\frac{q}{m_0}(\mathbf{g}^{\alpha\mu}F_{\mu\beta}-\mathbf{g}^{[\alpha\mu]}H_{\mu\beta})u^\beta-\frac{1}{8}\left(\frac{q}{m_0}\right)^2\tilde{\mathbf{g}}^{(\beta\alpha)}\left(\frac{1}{\rho^2}\right)_{,\beta}=0$$

$$\frac{q}{m_0}=\text{const} \tag{5.7}$$

Thus, we get a Lorentz force term in the equation of motion for a test particle. This term really differs from the analogous term in the Kaluza–Klein theory. If the metric is symmetric, we get the classical Lorentz force term. We also obtain an additional term

$$-\frac{1}{8}\left(\frac{q}{m_0}\right)^2\tilde{\mathbf{g}}^{(\beta\alpha)}\left(\frac{1}{\rho^2}\right)_{,\beta} \tag{5.8}$$

which expresses the interaction of the test particle with the scalar field ρ. If $\rho=\text{const}$, this term vanishes.

Equations (5.7) are defined on an electromagnetic bundle $\underline{P}$. The equations of motion for a test particle should be defined on E. This can be easily achieved by taking a local section of $\underline{P}$. For U(1) Abelian, $F_{\mu\nu}$ and $H_{\mu\nu}$ are well defined on E and we get exactly equation (5.7).

One can examine geodetic equations in a more geometrical way, i.e.,

$$\text{hor}(\nabla_u u)=\text{ver}(\nabla_u u)=0 \tag{5.9}$$

We get

$$\begin{aligned}&\text{hor}(\nabla_{\text{hor}(u(t))}\text{hor}(u(t))+\nabla_{\text{ver}(u(t))}\text{hor}(u(t))\\&\quad+\nabla_{\text{ver}(u(t))}\text{ver}(u(t))+\nabla_{\text{hor}(u(t))}\text{ver}(u(t)))=0\\&\text{ver}(\nabla_{\text{hor}(u(t))}\text{ver}(u(t))+\nabla_{\text{ver}(u(t))}\text{ver}(u(t))\\&\quad+\nabla_{\text{hor}(u(t))}\text{ver}(u(t))+\nabla_{\text{ver}(u(t))}\text{hor}(u(t)))=0\end{aligned} \tag{5.10}$$

Demanding that equation (5.10) possesses the first integral of motion

$$v=f(\text{ver}(u))=\text{const} \tag{5.11}$$

where f is a linear function of its argument, we get, according to the previous investigations,

$$\frac{dv}{dt}=0 \qquad \text{on} \quad \Gamma \tag{5.12}$$

and $f(y)=2\rho^2 y$.

Finally, it is of interest to mention that the exponential map on $(\underline{P}, \gamma)$, $\exp: T(\underline{P}) \to \underline{P}$ [$\exp_P: \mathrm{Tan}_P(\underline{P}) \to \underline{P}$ for each $p \in \underline{P}$, $\exp_P(v) = \Gamma_v(1)$, where $\Gamma_v(1)$ is an endpoint of a segment of a geodesic through p whose tangent at $\underline{P}$ is v for an arc parameter equal to 1], defines a normal coordinate system. Choosing an orthonormal basis $\{e_A\}$ for $\mathrm{Tan}_p(\underline{P})$, we define a coordinate system in a neighborhood of p by assigning to the point $\exp_P(\sum x^A e_A)$ the coordinates $(x^1, x^2, x^3, x^4, x^5)$. We call them normal coordinates. Usually in such a case one defines the function

$$\mp s^2 = (x^1)^2 - (x^2)^2 - (x^3)^2 - (x^4)^2 - (x^5)^2$$

The gradient of s is $\partial/\partial s$, where $\langle \mathrm{grad}\, f, X\rangle = df(X)$, $X \in T(P)$. Using this function, we can define the so-called polar coordinates s, θ_1, θ_2, θ_3, θ_4 (which are defined like the above normal coordinates). It is easy to see that the physical interpretation of the normal coordinates is as follows. They are the initial velocities and electric charges of test particles in such a way that $x^5 = (1/2\rho^2)(q/m_0)$ and $x^a = u_0^\alpha$. Since rays through the origin are geodesics, normal coordinates have the property that $\nabla_{\partial/\partial x^A}(\partial/\partial x^A) = 0$. Thus, $(\nabla_v \partial/\partial x^A)|_P = 0$. For these reasons normal coordinates are convenient to use. In the case of spacelike geodesics our interpretation breaks down, because they correspond to tachyon trajectories. Nonetheless, we can maintain this by supposing that u_0^α corresponds to the initial velocity of a tachyon as well.

6. GEOMETRY ON THE MANIFOLD $\underline{P}$

Let us turn to the calculation of the torsion for $\omega^A{}_B$,

$$\Theta^A(\Gamma) = D\theta^A$$

One easily gets

$$Q^\alpha{}_{\beta\gamma}(\Gamma) = \bar{Q}^\alpha_{\beta\gamma}(\bar{\Gamma}) \tag{6.1}$$

$$Q^\alpha{}_{5\beta}(\Gamma) = -Q^\alpha{}_{\beta 5}(\Gamma) = 2\rho^2(\mathbf{g}^{(\delta\alpha)}H_{\delta\beta} - \mathbf{g}^{\alpha\gamma}F_{\gamma\beta}) \tag{6.2}$$

$$Q^5{}_{\mu\nu}(\Gamma) = 2(F_{\mu\nu} - H_{\mu\nu}) \tag{6.3}$$

$$Q^5{}_{5\beta}(\Gamma) = -Q^5{}_{\beta 5}(\Gamma) = \frac{2}{\rho}\, \mathbf{g}_{[\beta\delta]}\tilde{\mathbf{g}}^{(\alpha\delta)}\rho_{,\alpha} \tag{6.4}$$

Let us define a tensor $K_{\beta\gamma}$ such that

$$H_{\beta\gamma} = F_{\beta\gamma} + K_{\beta\gamma} \tag{6.5}$$

Now we have

$$Q^5{}_{\beta\gamma}(\Gamma) = -2K_{\beta\gamma} \tag{6.6}$$

We find later a physical interpretation of this tensor. Now we calculate a 2-form of curvature for the connection $\omega^A{}_B$. We have

$$\Omega^A{}_B(\Gamma) = d\omega^A{}_B + \omega^A{}_C \wedge \omega^C{}_B \tag{6.7}$$

One easily gets, using (4.8) and (6.7),

$$\begin{aligned}\Omega^\alpha{}_\beta(\Gamma) = {}& \bar{\Omega}^\alpha{}_\beta(\bar{\Gamma}) + \rho^2[\mathbf{g}^{\delta\alpha} H_{\delta\beta} F_{\mu\nu} - \mathbf{g}^{\alpha\gamma}(H_{[\nu|\gamma|} - 2F_{[\nu|\gamma|})H_{|\beta|\omega]}]\theta^\mu \wedge \theta^\nu \\ & + [\bar{\nabla}_\mu(\rho^2 \mathbf{g}^{\delta\alpha} H_{\delta\beta}) + \rho H_{\beta\mu} \tilde{\mathbf{g}}^{(\gamma\alpha)} \rho_{,\gamma} \\ & + \rho \mathbf{g}^{\alpha\nu}(H_{\mu\nu} - 2F_{\mu\nu}) \mathbf{g}_{\beta\delta} \tilde{\mathbf{g}}^{(\alpha\delta)} \rho_{,\alpha}]\theta^\mu \wedge \theta^5\end{aligned} \tag{6.8a}$$

$$\begin{aligned}\Omega^5{}_\beta(\Gamma) = {}& \left[\bar{\nabla}_{[\mu} H_{\nu]\beta} + \frac{1}{2} H_{\gamma\beta} \bar{Q}^\gamma{}_{\mu\nu}(\Gamma) + \frac{1}{\rho} \mathbf{g}_{\beta\delta} \tilde{\mathbf{g}}^{(\alpha\delta)} \rho_{,\alpha} F_{\mu\nu}\right. \\ & \left. + \frac{1}{\rho} \mathbf{g}_{\delta[\mu} \tilde{\mathbf{g}}^{(\alpha\delta)} \rho_{,|\alpha|} H_{|\beta|\nu]}\right]\theta^\mu \wedge \theta^\nu \\ & + \left[\bar{\nabla}_\mu\left(\frac{1}{\rho} \mathbf{g}_{\beta\gamma} \tilde{\mathbf{g}}^{(\alpha\delta)} \rho_{,\alpha}\right) + \rho^2 \mathbf{g}^{\delta\gamma} H_{\delta\beta} H_{\gamma\mu}\right. \\ & \left. + \frac{1}{\rho^2} \mathbf{g}_{\delta\mu} \mathbf{g}_{\beta\gamma} \tilde{\mathbf{g}}^{(\alpha\delta)} \tilde{\mathbf{g}}^{(\nu\gamma)} \rho_{,\alpha} \rho_{,\nu}\right]\theta^\mu \wedge \theta^5\end{aligned} \tag{6.8b}$$

$$\begin{aligned}\Omega^\alpha{}_5(\Gamma) = {}& \{\bar{\nabla}_{[\mu}[\rho^2 \mathbf{g}^{\alpha\beta}(H_{\nu]\beta} - 2F_{\nu]\beta})] + \tfrac{1}{2}\rho^2 \mathbf{g}^{\alpha\beta}(H_{\gamma\beta} - 2F_{\gamma\beta})\bar{Q}^\gamma_{\mu\nu}(\bar{\Gamma}) \\ & + \rho \tilde{\mathbf{g}}^{(\gamma\alpha)} \rho_{,\gamma} F_{\mu\nu} + \rho \mathbf{g}^{\alpha\beta} \mathbf{g}_{\delta[\nu} \tilde{\mathbf{g}}^{(\alpha\delta)} \rho_{,|\alpha|}(H_{\mu]\beta} - 2F_{\mu]\beta})\}\theta^\mu \wedge \theta^\nu \\ & + [\bar{\nabla}_\mu(\rho \tilde{\mathbf{g}}^{(\gamma\alpha)} \rho_{,\gamma}) + \rho^4 \mathbf{g}^{\delta\alpha} \mathbf{g}^{\gamma\beta} H_{\delta\gamma}(H_{\beta\mu} - 2F_{\beta\mu}) \\ & - \mathbf{g}_{\delta\mu} \tilde{\mathbf{g}}^{(\gamma\alpha)} \tilde{\mathbf{g}}^{(\nu\delta)} \rho_{,\gamma} \rho_{,\nu}]\theta^\mu \wedge \theta^5\end{aligned} \tag{6.8c}$$

$$\begin{aligned}\Omega^5{}_5(\Gamma) = {}& \left[\bar{\nabla}_{[\mu}\left(\frac{1}{\rho} \mathbf{g}_{|\delta|\nu]} \tilde{\mathbf{g}}^{(\alpha\delta)} \rho_{,\alpha}\right) + \frac{1}{\rho} \mathbf{g}_{\delta\gamma} \tilde{g}^{(\alpha\delta)} \rho_{,\alpha} \bar{Q}^\gamma_{\mu\nu}(\bar{\Gamma})\right. \\ & \left. - \rho^2 H_{\beta[\nu} \mathbf{g}^{\beta\alpha}(H_{\mu]\alpha} - 2F_{\mu]\alpha}) + \frac{1}{\rho^2} \mathbf{g}_{\delta[\mu} \mathbf{g}_{|\gamma|\nu]} \tilde{\mathbf{g}}^{(\alpha\delta)} \tilde{\mathbf{g}}^{(\beta\gamma)} \rho_{,\alpha} \rho_{,\beta}\right]\theta^\mu \wedge \theta^\nu \\ & + [\rho \tilde{\mathbf{g}}^{(\gamma\beta)} \rho_{,\gamma} H_{\beta\mu} - \rho \mathbf{g}^{(\alpha\delta)} \rho_{,\alpha}(H_{\mu\delta} - 2F_{\mu\delta})]\theta^5 \wedge \theta^\mu\end{aligned} \tag{6.8d}$$

where $\bar{\Omega}^{\alpha}{}_{\beta}(\Gamma)$ is the 2-form of curvature of the connection $\bar{\omega}^{\alpha}{}_{\beta}$, is the covariant derivative with respect to $\bar{\omega}^{\alpha}{}_{\beta}$, and $\bar{Q}^{\alpha}{}_{\beta\gamma}(\bar{\Gamma})$ is the tensor of torsion for $\bar{\omega}^{\alpha}{}_{\beta}$. One easily reads from (6.8a)–(6.8d) the tensor of curvature for $\omega^{A}{}_{B}$. We have

$$R^{\alpha}{}_{\beta\mu\nu}=\bar{R}^{\alpha}{}_{\beta\mu\nu}+2\rho^{2}[\mathbf{g}^{\delta\alpha}H_{\delta\beta}F_{\mu\nu}-\mathbf{g}^{\alpha\gamma}(H_{[\nu|\gamma|}-2F_{[\nu|\gamma|})H_{|\beta|\mu]}] \tag{6.9a}$$

$$\begin{aligned}R^{\alpha}{}_{\beta\mu5}&=-R^{\alpha}{}_{\beta5\mu}\\&=\bar{\nabla}_{\mu}(\rho^{2}\mathbf{g}^{\delta\alpha}H_{\delta\beta})+\rho H_{\beta\mu}\tilde{\mathbf{g}}^{(\gamma\alpha)}\rho_{,\gamma}+\rho\mathbf{g}^{\alpha\nu}(H_{\mu\nu}-2F_{\mu\nu})\mathbf{g}_{\beta\delta}\tilde{\mathbf{g}}^{(\alpha\delta)}\rho_{,\alpha}\end{aligned} \tag{6.9b}$$

$$\begin{aligned}R^{5}{}_{\beta\mu\nu}&=2\bar{\nabla}_{[\mu}H_{\nu]\beta}+H_{\gamma\beta}\bar{Q}^{\gamma}{}_{\mu\nu}(\bar{\Gamma})\\&\quad+\frac{2}{\rho}\tilde{\mathbf{g}}^{(\alpha\delta)}\rho_{,\alpha}F_{\mu\nu}+\frac{2}{\rho}\mathbf{g}_{\delta[\mu}\tilde{\mathbf{g}}^{(\alpha\delta)}\rho_{,|\alpha|}H_{|\beta|\nu]}\end{aligned} \tag{6.9c}$$

$$\begin{aligned}R^{5}{}_{\beta\mu5}&=-R^{5}{}_{\beta5\mu}\\&=\bar{\nabla}_{\mu}\left(\frac{2}{\rho}\mathbf{g}_{\beta\delta}\tilde{\mathbf{g}}^{(\alpha\delta)}\rho_{,\alpha}\right)+\rho^{2}\mathbf{g}^{\delta\gamma}H_{\delta\beta}H_{\gamma\mu}+\frac{1}{\rho^{2}}\mathbf{g}_{\delta\mu}\tilde{\mathbf{g}}^{(\alpha\delta)}\tilde{\mathbf{g}}^{(\nu\gamma)}\rho_{,\alpha}\rho_{,\nu}\end{aligned} \tag{6.9d}$$

$$\begin{aligned}R^{\alpha}{}_{5\mu\nu}&=2\bar{\nabla}_{[\mu}[\rho^{2}\mathbf{g}^{\alpha\beta}(H_{\nu]\beta}-2F_{\nu]\beta})]\\&\quad+\rho^{2}\mathbf{g}^{\alpha\beta}(H_{\gamma\beta}-2F_{\gamma\beta})\bar{Q}^{\gamma}{}_{\mu\nu}(\bar{\Gamma})+2\rho\tilde{\mathbf{g}}^{(\gamma\alpha)}\rho_{,\gamma}F_{\mu\nu}\\&\quad+2\rho\mathbf{g}^{\alpha\beta}\mathbf{g}_{\delta[\nu}\tilde{\mathbf{g}}^{(\alpha\delta)}\rho_{,|\alpha|}(H_{\mu]\beta}-2F_{\mu]\beta})\end{aligned} \tag{6.9e}$$

$$\begin{aligned}R^{\alpha}{}_{5\mu5}&=-R^{\alpha}{}_{55\mu}\\&=\bar{\nabla}_{\mu}(2\rho\tilde{\mathbf{g}}^{(\gamma\alpha)}\rho_{,\gamma})+\rho^{4}\mathbf{g}^{\delta\alpha}\mathbf{g}^{\gamma\beta}H_{\delta\gamma}(H_{\beta\mu}-2F_{\beta\mu})-\mathbf{g}_{\delta\mu}\tilde{\mathbf{g}}^{(\gamma\alpha)}\tilde{\mathbf{g}}^{(\nu\delta)}\rho_{,\gamma}\rho_{,\nu}\end{aligned} \tag{6.9f}$$

$$\begin{aligned}R^{5}{}_{5\mu\nu}&=2\bar{\nabla}_{[\mu}\left(\frac{1}{\rho}\mathbf{g}_{|\delta|\nu]}\tilde{\mathbf{g}}^{(\alpha\delta)}\rho_{|\alpha}\right)+\frac{1}{\rho}\mathbf{g}_{\delta\gamma}\tilde{\mathbf{g}}^{(\alpha\delta)}\rho_{,\alpha}\bar{Q}^{\gamma}{}_{\mu\nu}(\Gamma)\\&\quad-2\rho^{2}\mathbf{g}^{\beta\alpha}H_{\beta[\nu}(H_{\mu]\alpha}-2F_{\mu]\alpha})+\frac{2}{\rho^{2}}\mathbf{g}_{\delta[\mu}\tilde{\mathbf{g}}_{|\gamma|\nu]}\tilde{\mathbf{g}}^{(\alpha\delta)}\tilde{\mathbf{g}}^{(\beta\gamma)}\rho_{|\alpha}\rho_{|\beta}\end{aligned} \tag{6.9g}$$

$$R^{5}{}_{55\mu}=-R^{5}{}_{5\mu5}=-\rho\tilde{\mathbf{g}}^{(\alpha\delta)}\rho_{|\alpha}(H_{\mu\delta}-2F_{\mu\delta})+\rho\tilde{\mathbf{g}}^{(\gamma\beta)}\rho_{,\gamma}H_{\beta\mu} \tag{6.9h}$$

In Section 4 we defined the connection $W^{A}{}_{B}$ in terms of $\omega^{A}{}_{B}$. Let us calculate the two-form of curvature for $W^{A}{}_{B}$,

$$\Omega^{A}{}_{B}(W)=dW^{A}{}_{B}+W^{A}{}_{C}\wedge W^{C}{}_{B} \tag{6.10}$$

We have

$$\Omega^{A}{}_{B}(W)=\Omega^{A}{}_{B}(\Gamma)-\tfrac{4}{9}\delta^{A}_{B}\bar{W}_{[\mu,\nu]}\theta^{\mu}\wedge\theta^{\nu} \tag{6.11}$$

One easily finds the relations between the curvature tensors for $W^A{}_B$ and $\omega^A{}_B$.

$$R^\alpha{}_{\beta\mu\nu}(W)=R^\alpha{}_{\beta\mu\nu}(\Gamma)-\tfrac{8}{9}\delta^\alpha_\beta W_{[\mu,\nu]} \tag{6.12a}$$

$$R^5{}_{5\mu\nu}(W)=R^5{}_{5\mu\nu}(\Gamma)-\tfrac{8}{9}W_{[\mu,\nu]} \tag{6.12b}$$

$$R^\alpha{}_{\beta 5\nu}(W)=R^\alpha{}_{\beta 5\nu}(\Gamma) \tag{6.12c}$$

$$R^5{}_{\beta\mu 5}(W)=R^5{}_{\beta\mu 5}(\Gamma) \tag{6.12d}$$

Now we pass to the calculation of the contraction of $R^A{}_{BCD}(\Gamma)$,

$$A_{BC}(\Gamma)=R^A{}_{BCA}(\Gamma) \tag{6.13}$$

and the Ricci scalar

$$A(\Gamma)=\gamma^{AC}A_{AB}(\Gamma)=\mathbf{g}^{\beta\gamma}A_{\beta\gamma}(\Gamma)-\frac{1}{\rho^2}A_{55} \tag{6.14}$$

After some calculations one easily gets

$$A(\Gamma)=\bar{A}(\bar{\Gamma})+\rho^2[(\mathbf{g}^{[\mu\nu]}F_{\mu\nu})^2-H^{\alpha\mu}F_{\alpha\mu}] + \mathbf{g}^{\beta\mu}\left[\bar{\nabla}_\mu\left(\frac{1}{\rho}\mathbf{g}_{\beta\delta}\tilde{\mathbf{g}}^{(\alpha\delta)}\rho_{|\alpha}\right)\right]+\frac{1}{\rho^2}\bar{\nabla}_\alpha(\rho\tilde{\mathbf{g}}^{(\gamma\alpha)}\rho_{,\gamma}) \tag{6.15}$$

Now we can calculate the Moffat–Ricci curvature scalar for the connection $\omega^A{}_B$:

$$R(\Gamma)=\gamma^{BC}[R^A{}_{BCA}(\Gamma)+\tfrac{1}{2}R^A{}_{ABC}(\Gamma)] \tag{6.16}$$

One easily gets after some calculations

$$\sqrt{\gamma}\,R(\Gamma)=\sqrt{-\mathbf{g}}\,\rho\left\{\bar{R}(\bar{\Gamma})+\rho^2[2(\mathbf{g}^{[\mu\nu]}F_{\mu\nu})^2-H^{\alpha\mu}F_{\alpha\mu}] + \frac{1}{\rho^2}\mathbf{g}^{[\nu\mu]}\mathbf{g}_{\delta\nu}\tilde{\mathbf{g}}^{(\alpha\delta)}\rho_{,\mu}\rho_{,\alpha}\right\}+\partial_\mu K^\mu \tag{6.17}$$

where

$$K^\mu=\tfrac{1}{2}\sqrt{-\mathbf{g}}\,\rho_{,\gamma}(5\tilde{\mathbf{g}}^{(\gamma\mu)}-\mathbf{g}^{\nu\mu}\mathbf{g}_{\gamma\nu}\tilde{\mathbf{g}}^{(\gamma\delta)})$$

$\bar{R}(\bar{\Gamma})$ is the Moffat–Ricci curvature scalar for the connection $\bar{\omega}^\alpha{}_\beta$ on E,

$$H^{\alpha\mu}=\mathbf{g}^{\beta\mu}\mathbf{g}^{\gamma\alpha}H_{\beta\gamma},\qquad H_{\beta\gamma}=-H_{\gamma\beta} \tag{6.18}$$

and

$$\mathrm{g}_{\delta\beta}\mathrm{g}^{\gamma\delta}H_{\gamma\alpha}+\mathrm{g}_{\alpha\delta}\mathrm{g}^{\delta\gamma}H_{\beta\gamma}=2\mathrm{g}_{\alpha\delta}\mathrm{g}^{\delta\gamma}F_{\beta\gamma} \tag{6.19}$$

Now we are able to write $\sqrt{\gamma}\,R(W)$, where $R(W)$ is the Moffat–Ricci curvature scalar for the connection $W^A{}_B$ on $\underline{P}$, and we easily find

$$\sqrt{\gamma}\,R(W)=\sqrt{-\mathrm{g}}\,\rho\left\{\bar{R}(\bar{W})+\rho^2[2(\mathrm{g}^{[\mu\nu]}F_{\mu\nu})^2-H^{\alpha\mu}F_{\alpha\mu}]+\frac{1}{\rho^2}\mathrm{g}^{[\nu\mu]}\mathrm{g}_{\delta\nu}\tilde{\mathrm{g}}^{(\alpha\delta)}\rho_{,\mu}\rho_{,\alpha}\right\}+\partial_\mu K^\mu \tag{6.20}$$

where $\bar{R}(\bar{W})$ is the Moffat–Ricci curvature scalar for the connection $\bar{W}^\alpha{}_\beta$ on E. It is easy to see that from the variational principle point of view it is enough to consider in the place of $\sqrt{\gamma}\,R(W)$ only

$$\sqrt{\gamma}\,B(W)=\sqrt{-\mathrm{g}}\,\rho\left\{\bar{R}(\bar{W})+\rho^2[2(\mathrm{g}^{[\mu\nu]}F_{\mu\nu})^2-H^{\alpha\mu}F_{\alpha\mu}]+\frac{1}{\rho^2}\mathrm{g}^{[\nu\mu]}\mathrm{g}_{\delta\nu}\tilde{\mathrm{g}}^{(\alpha\delta)}\rho_{,\mu}\rho_{,\alpha}\right\} \tag{6.21}$$

The four-divergence $\partial_\mu K^\mu$ (an exact form) plays role in topological considerations.

Finally let us note some identities for $F_{\mu\nu}$ and $H_{\mu\nu}$. One gets from equation (1.9).

$$\mathrm{g}^{[\mu\nu]}H_{\mu\nu}=\mathrm{g}^{[\mu\nu]}F_{\mu\nu} \tag{6.22}$$

$$\mathrm{g}^{\alpha\omega}\mathrm{g}^{\beta\mu}H_{\alpha\beta}H_{\omega\mu}=\mathrm{g}^{\alpha\omega}\mathrm{g}^{\beta\mu}H_{\alpha\beta}F_{\omega\mu} \tag{6.23}$$

$$\mathrm{g}^{\sigma\nu}\mathrm{g}^{\alpha\mu}H_{\sigma\alpha}F_{\mu\nu}+\mathrm{g}^{\mu\sigma}\mathrm{g}^{\nu\beta}H_{\beta\sigma}F_{\mu\nu}=2\mathrm{g}^{\mu\sigma}\mathrm{g}^{\nu\beta}F_{\mu\nu}F_{\beta\sigma} \tag{6.24}$$

7. CONFORMAL TRANSFORMATION OF $\mathrm{g}_{\mu\nu}$. TRANSFORMATION OF THE SCALAR FIELD ρ

In Section 3.4 we get the Moffat–Ricci curvature scalar for the connection $W^A{}_B$ on $\underline{P}$. The appropriate scalar density on $\underline{P}$ differs from $\sqrt{\gamma}\,B(W)$ [see equation (6.21)] only by the exact form (full divergence for the vector K^μ). We will consider $\sqrt{\gamma}\,B(W)$ as the Lagrangian density for the gravitational, electromagnetic, and scalar fields. Bergmann[50] considers a general

Lagrangian of this type (see Ref. 50, p. 26, formula [1.1]). Our Lagrangian has of course the determined functions f_1, f_2, f_3, f_4. It is easy to see that

$$f_1(\rho)=\rho, \qquad f_2(\rho)=\rho^3, \qquad f_3(\rho)=\frac{1}{\rho}, \qquad f_4(\rho)=0 \tag{7.1}$$

In equation (6.21) we get also a special form of the Lagrangian for the scalar field ρ. It is

$$\mathscr{L}(\rho)=\frac{1}{\rho^2}\,\underset{\sim}{\mathbf{g}}^{[\nu\mu]}\mathbf{g}_{\delta\nu}\tilde{\mathbf{g}}^{(\alpha\delta)}\rho_{,\mu}\rho_{,\alpha} \tag{7.2}$$

It vanishes if the skew-symmetric part of the metric is zero. Thus, the scalar field will propagate if the skew-symmetric part of the metric is not zero.

Let us transform ρ and $\mathbf{g}_{\mu\nu}$ in the way suggested in Ref. 50:

$$\rho=e^{-\Psi} \tag{7.3}$$

$$\mathbf{g}_{\mu\nu}\to e^{\Psi}\mathbf{g}_{\mu\nu}=\frac{1}{\rho}\,\mathbf{g}_{\mu\nu} \tag{7.4}$$

(conformal transformation of $\mathbf{g}_{\mu\nu}$). After simple calculations one obtains

$$\begin{aligned}\sqrt{\gamma}\,B(W)=\sqrt{-\mathbf{g}}\,\{&\bar{R}(\bar{W})+e^{-3\Psi}[2(\mathbf{g}^{[\mu\nu]}F_{\mu\nu})^2-H^{\alpha\mu}F_{\alpha\mu}]\\&+\mathbf{g}^{[\mu\nu]}\mathbf{g}_{\delta\nu}\tilde{\mathbf{g}}^{(\alpha\delta)}\Psi_{,\mu}\Psi_{,\alpha}\}\end{aligned} \tag{7.5}$$

Thus, we have the Lagrangian density in our theory

$$L(\bar{W},\mathbf{g}_{\mu\nu},A_\mu,\Psi)=\sqrt{-\mathbf{g}}\;\bar{R}(W)+8\pi e^{-3\Psi}L_{\text{em}}+L_{\text{scal}}(\Psi) \tag{7.6}$$

where

$$L_{\text{em}}=\frac{\sqrt{-\mathbf{g}}}{8\pi}\,[2(\mathbf{g}^{[\mu\nu]}F_{\mu\nu})^2-H^{\alpha\mu}F_{\alpha\mu}] \tag{7.7}$$

$$L_{\text{scal}}=\underset{\sim}{\mathbf{g}}^{[\nu\mu]}\mathbf{g}_{\delta\nu}\tilde{\mathbf{g}}^{(\alpha\delta)}\Psi_{,\mu}\Psi_{,\alpha} \tag{7.8}$$

and

$$\underset{\sim}{\mathbf{g}}^{[\nu\mu]}=\sqrt{-\mathbf{g}}\;\mathbf{g}^{[\nu\mu]} \tag{7.9}$$

This is of course a scalar–tensor theory of gravitation with a nonsymmetric metric unified with electromagnetism. The scalar field propagates only if the skew-symmetric part of the metric is not zero. Otherwise, Ψ is not a dynamical field. In the next section we will find an interpretation of this field.

8. THE VARIATIONAL PRINCIPLE AND FIELD EQUATIONS—INTERPRETATIONS AND CONCLUSIONS

Let us define the Palatini variational principle on the manifold $\underline{P}$ for $R(W)$,

$$\delta \int_V R(W)\sqrt{\gamma}\, d^5X = 0, \qquad V \subset \underline{P} \tag{8.1}$$

where

$$\gamma = \det(\gamma_{AB}) = -\rho^2 \det(\mathbf{g}_{\alpha\beta}) = -\mathbf{g}\rho^2$$

It is easy to see that (8.1) is equivalent to the following:

$$\delta \int_{\mathrm{U}} \sqrt{-\mathbf{g}}\, d^4X\{\bar{R}(\bar{W}) + e^{-3\Psi}[2(\mathbf{g}^{[\mu\nu]}F_{\mu\nu})^2 - H^{\alpha\mu}F_{\alpha\mu}] + \mathbf{g}^{[\nu\mu]}\mathbf{g}_{\delta\nu}\tilde{\mathbf{g}}^{(\alpha\delta)}\Psi_{,\mu}\Psi_{,\alpha}\} = 0 \tag{8.2}$$

where $\mathrm{U} \subset E$. We vary with respect to the independent quantities $\bar{W}^{\alpha}{}_{\beta\gamma}$, $\mathbf{g}_{\mu\nu}$, A_μ, and Ψ. After some calculations we easily get

$$\bar{R}_{\alpha\beta}(\bar{W}) - \tfrac{1}{2}\mathbf{g}_{\alpha\beta}\bar{R}(\bar{W}) = 8\pi K[\overset{\text{em}}{T}_{\alpha\beta} + \overset{\text{scal}}{T}_{\alpha\beta}(\Psi)] \tag{8.3}$$

$$\mathbf{g}_{\mu\nu\,\alpha} - \mathbf{g}_{\xi\nu}\bar{\Gamma}^{\xi}{}_{\mu\sigma} - \mathbf{g}_{\mu\xi}\bar{\Gamma}^{\xi}{}_{\sigma\nu} = 0 \tag{8.4}$$

$$\mathbf{g}^{[\mu\nu]}{}_{,\nu} = 0 \tag{8.5}$$

We can rewrite equation (8.5) in the form

$$\bar{\nabla}_\nu \mathbf{g}^{[\mu\nu]} = 0 \tag{8.5a}$$

$$\partial_\mu(\mathbf{H}^{\alpha\mu}) = 2\mathbf{g}^{[\alpha\beta]}\partial_\beta(\mathbf{g}^{[\mu\nu]}F_{\mu\nu}) - 3\partial_\beta\Psi(\mathbf{H}^{\beta\alpha} - 2\mathbf{g}^{[\beta\alpha]}(\mathbf{g}^{[\mu\nu]}F_{\mu\nu})) \tag{8.6}$$

We can write in place of $\partial_\mu(\mathbf{H}^{\alpha\mu})$, $\sqrt{\mathbf{g}}\cdot\bar{\nabla}_\mu H^{\alpha\mu}$, and

$$(\tilde{\mathbf{g}}^{(\alpha\mu)} - \mathbf{g}^{\nu\mu}\mathbf{g}_{\delta\nu}\tilde{\mathbf{g}}^{(\alpha\delta)})\frac{\partial^2\Psi}{\partial x^\alpha\,\partial x^\mu} + \frac{1}{\sqrt{-\mathbf{g}}}\partial_\mu\{\sqrt{-\mathbf{g}}\,[\tilde{\mathbf{g}}^{(\mu\alpha)} - \tfrac{1}{2}\mathbf{g}_{\delta\nu}(\mathbf{g}^{\nu\alpha}\tilde{\mathbf{g}}^{(\mu\delta)} + \mathbf{g}^{\nu\mu}\tilde{\mathbf{g}}^{(\alpha\delta)})]\}\frac{\partial\Psi}{\partial x^\alpha} - \frac{6\pi}{\sqrt{-\mathbf{g}}}e^{-3\Psi}\mathscr{L}_{\text{em}} = 0 \tag{8.7}$$

where

$$\overset{\text{em}}{T}_{\alpha\beta} = \frac{1}{4\pi}\{\mathbf{g}_{\gamma\beta}\mathbf{g}^{\tau\mu}\mathbf{g}^{\varepsilon\gamma}H_{\mu\alpha}H_{\tau\varepsilon} - 2\mathbf{g}^{[\mu\nu]}F_{\mu\nu}F_{\alpha\beta}$$
$$- \tfrac{1}{4}\mathbf{g}_{\alpha\beta}[H^{\mu\nu}F_{\mu\nu} - 2(\mathbf{g}^{[\mu\nu]}F_{\mu\nu})^2]\} \tag{8.8}$$

is the energy-momentum tensor for the electromagnetic field in the non-symmetric Kaluza–Klein theory

$$\overset{\text{scal}}{T}_{\alpha\beta}(\Psi) = \frac{e^{3\Psi}}{4\pi}$$
$$\times [\tfrac{1}{2}\tilde{\mathbf{g}}^{(\xi\kappa)}\tilde{\mathbf{g}}^{(\omega\delta)}(\mathbf{g}_{\kappa\beta}\mathbf{g}_{\omega\alpha} + \mathbf{g}_{\omega\beta}\mathbf{g}_{\kappa\alpha})(\mathbf{g}^{\nu\mu}\mathbf{g}_{\delta\nu} - \delta^{\mu}_{\delta})\Psi_{,\mu}\Psi_{,\xi}$$
$$- \mathbf{g}_{\alpha\beta}(\mathbf{g}^{[\nu\mu]}\mathbf{g}_{\delta\nu}\tilde{\mathbf{g}}^{(\omega\delta)}\Psi_{,\mu}\Psi_{,\omega})] \tag{8.9}$$

is the energy-momentum tensor for the scalar field Ψ. In the calculations we used the following formulas for $\delta\mathbf{g}_{\gamma\nu}$ and $\delta\tilde{\mathbf{g}}^{(\alpha\delta)}$:

$$\delta\mathbf{g}_{\delta\nu} = -\mathbf{g}_{\nu\gamma}\mathbf{g}_{\delta\omega}\,\delta\mathbf{g}^{\omega\gamma} \tag{8.10}$$

$$\delta\tilde{\mathbf{g}}^{(\alpha\delta)} = \tfrac{1}{2}(\tilde{\mathbf{g}}^{(\alpha\beta)}\tilde{\mathbf{g}}^{(\omega\delta)}\mathbf{g}_{\beta\gamma}\mathbf{g}_{\omega\psi} + \tilde{\mathbf{g}}^{(\omega\delta)}\tilde{\mathbf{g}}^{(\alpha\beta)}\mathbf{g}_{\omega\gamma}\mathbf{g}_{\beta\psi})\,\delta\mathbf{g}^{\psi\gamma} \tag{8.11}$$

It is easy to see that the trace of $\overset{\text{scal}}{T}_{\alpha\beta}(\Psi)$ is not zero,

$$g^{\alpha\beta}\overset{\text{scal}}{T}_{\alpha\beta}(\Psi) = -e^{3\Psi}/8\pi(\mathbf{g}^{[\nu\mu]}\mathbf{g}_{\delta\nu}\tilde{\mathbf{g}}^{(\omega\delta)}\Psi_{,\mu}\Psi_{,\omega}) \neq 0 \tag{8.12}$$

We have

$$\mathbf{H}^{\alpha\mu} = \sqrt{-\mathbf{g}}\mathbf{g}^{\beta\mu}\mathbf{g}^{\gamma\alpha}H_{\beta\gamma}, \qquad H_{\beta\gamma} = -H_{\gamma\beta} \tag{8.13}$$

and

$$\mathbf{g}_{\delta\beta}\mathbf{g}^{\gamma\delta}H_{\gamma\alpha} + \mathbf{g}_{\alpha\delta}\mathbf{g}^{\delta\gamma}H_{\beta\gamma} = 2\mathbf{g}_{\alpha\delta}\mathbf{g}^{\delta\gamma}F_{\beta\gamma} \tag{8.14}$$

We can rewrite equation (8.14) in a matrix notation,

$$\mathbf{g}(\mathbf{g}^{-1})^T H + \mathbf{g}^T\mathbf{g}^{-1}H^T = 2\mathbf{g}^T\mathbf{g}^{-1}F \tag{8.14a}$$

where T means a matrix transposition. Equation (8.14) expresses the relationship between tensors $H_{\alpha\beta}$ and $F_{\alpha\beta}$.

It is well known from Einstein's unified field theory that equation (8.4) has the following solution:

$$\bar{\Gamma}^{\alpha}{}_{\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\} + \tfrac{1}{2}\bar{Q}^{\alpha}{}_{\beta\gamma} + U^{\alpha}{}_{\beta\gamma}$$

where $\bar{Q}^{\alpha}{}_{\beta\gamma}$ is the torsion of the connection $\bar{\Gamma}$, $\{{}^{\alpha}_{\beta\gamma}\}$ is the Christoffel symbol form for $\mathbf{g}_{(\alpha\beta)}$, and

$$U^{\nu}{}_{\lambda\mu} = \tilde{\mathbf{g}}^{(\nu\alpha)} \bar{Q}_{\alpha(\lambda}\beta \mathbf{g}_{[\mu)\beta]}$$

Now we are able to interpret the quantities in our theory. First of all it is easy to see that $H_{\alpha\beta}$ plays the role of the second tensor of the electromagnetic strength (the so-called induction tensor) and equation (8.14) expresses the relationship between tensors $F_{\alpha\beta}$ and $H_{\alpha\beta}$.

In the classical electrodynamics of continuous media[68] or in nonlinear electrodynamics[69] it is necessary to define both of these tensors. The first tensor, $F_{\alpha\beta}$, is built from (**E**, **B**) and the second, $H_{\alpha\beta}$, from (**D**, **H**).

If the metric $\mathbf{g}_{\alpha\beta}$ is symmetric, then $F_{\alpha\beta} = H_{\alpha\beta}$. Thus, it is interesting that the skew-symmetric part of the metric $\mathbf{g}_{[\alpha\beta]}$ induces some kind of electromagnetic polarization tensor of the vacuum.

In the classical electrodynamics of continuous media[66] and in nonlinear electrodynamics[67] it is possible to define the electromagnetic polarization tensor of the continuous medium (classical electrodynamics) or the vacuum (nonlinear electrodynamics) called $M_{\alpha\beta}$,

$$H_{\alpha\beta} = F_{\alpha\beta} - 4\pi M_{\alpha\beta} \tag{8.15}$$

It is easy to see that

$$4\pi M_{\alpha\beta} = -K_{\alpha\beta} \tag{8.16}$$

[see (6.6)]. Thus, we get a geometrical interpretation of $M_{\alpha\beta}$.

$$Q^{5}{}_{\alpha\beta}(\Gamma) = 8\pi M_{\alpha\beta} \tag{8.17}$$

The electromagnetic polarization induced by the skew-symmetric part of the metric $\mathbf{g}_{[\mu\nu]}$ is the torsion in the fifth dimension. This is in very good accordance with results from Refs. 16 and 68. The only difference is that in Refs. 16 and 68 the electromagnetic polarization has its origin in external sources and (8.17) plays the role of the Cartan equation in the Kaluza–Klein theory with torsion.

But this is not all. The skew-symmetric part of the metric $\mathbf{g}_{[\mu\nu]}$ also changes the electromagnetic Lagrangian.

$$\mathscr{L}_{\text{em}} = \frac{1}{8\pi} [2(\mathbf{g}^{[\mu\nu]} F_{\mu\nu})^2 - H^{\mu\alpha} F_{\mu\alpha}] \tag{8.18}$$

In (8.18) we have a new term $2(g^{[\mu\nu]} F_{\mu\nu})^2$ which is an interaction between the skewon field and the electromagnetic field. This term vanishes if the metric is symmetric and is always nonnegative.

Thus, classical electrodynamics in the nonsymmetric Kaluza–Klein theory will be different than in general relativity.

The skew-symmetric part of the metric also induces a source for the electromagnetic field. In equation (8.6) we get a current.

$$\underset{\sim}{j}^{\alpha} = -\frac{1}{2\pi} \mathbf{g}^{[\alpha\beta]} \, \partial_\beta (\mathbf{g}^{[\mu\nu]} F_{\mu\nu}) \tag{8.19}$$

which is conserved automatically *modulo* equation (8.5)

$$\underset{\sim}{j}^{\alpha}{}_{,\alpha} = 0 \tag{8.20}$$

This current vanishes if the metric is symmetric. This is completely different from the classical Kaluza–Klein theory.[1,16] In the classical approach based on a symmetric metric on space-time one obtains the second Maxwell equation in the vacuum. There is also an additional current induced by a scalar field Ψ. We have for K

$$K = e^{-3\Psi} \tag{8.21}$$

Equation (8.21) expresses the relation between the scalar field Ψ and the gravitational constant K. Simultaneously we get an interpretation of the scalar field Ψ. It is connected to the gravitational "constant" K, which now can change in space and time according to equations (8.21) and (8.7). It is easy to see that if the symmetric part of the metric is zero, K is not a dynamical field and it is really a constant. In this way the scalar field ρ is also a constant. Thus, we get zero for the extra term in equation (5.7). In this way equation (5.7) becomes the ordinary equation of motion for a charged test particle in gravitational and electromagnetic fields. It is easy to see that equation (8.7) is more similar to the Klein–Gordon equation than to the wave equation: consider the term

$$\frac{6\pi}{\sqrt{-\mathbf{g}}} e^{-3\Psi} \mathscr{L}_{\mathrm{em}} \tag{8.22}$$

Only if

$$\mathscr{L}_{\mathrm{em}} = 0 \tag{8.23}$$

do we get an analogue of the wave equation. It is equivalent to

$$2(\mathbf{g}^{[\mu\nu]} F_{\mu\nu})^2 = H^{\alpha\mu} F_{\alpha\nu} \tag{8.24}$$

Thus, we get the equations for the gravitational field in the tensor–scalar theory with electromagnetic and scalar sources. Moreover, the tensorial part of the gravitational potentials is not symmetric. These are equations (8.3) and (8.5). Equation (8.4) is a compatibility condition from Moffat's theory of gravitation.

Now we turn to the problem of the equivalence principle in our theory. It is well known[50] that many scalar–tensor theories of gravitation do not satisfy the equivalence principle. They are in contradiction with the universal free fall of all bodies. This occurs of course iff scalar forces have long range. In this case the scalar field satisfies the wave equation. But fortunately this is not true in our case. Equation (8.7) is of the Klein–Gordon type rather than of the wave type. This suggests that our scalar forces obey Yukawa-type behavior and not Coulomb behavior. This means that our scalar forces are of short distance, i.e.,

$$\Psi \sim \frac{1}{r} e^{-\alpha r}, \qquad \alpha > 0 \tag{8.25}$$

This means that if $r \to \infty$, $\Psi \to 0$ and

$$K \to \text{const} \tag{8.26}$$

Thus, we get the unification of Moffat's theory of gravitation with electromagnetism and scalar theory. This nonsymmetric version of the Jordan–Thiry theory combines gravitational theory and the electromagnetic Maxwell theory in a much stronger way than the classical Kaluza–Klein theory and the classical Jordan–Thiry theory. Simultaneously we get the possibility for changing the gravitational constant without violation of the equivalence principle. In our approach there exist "interference effects" between gravitation and electromagnetism which are absent in the classical approaches.

1. A new term in the electromagnetic Lagrangian

$$\frac{1}{4\pi} (\mathbf{g}^{[\mu\nu]} F_{\mu\nu})^2$$

2. The existence of an electromagnetic polarization of the vacuum M which has a geometrical interpretation as a torsion in the fifth dimension
3. An additional term for the Lorentz force term in the equation of motion for a test particle

$$\frac{q}{m_0} \mathbf{g}^{[\gamma\alpha]} H_{\gamma\beta} u^\beta$$

4. A new energy-momentum tensor $\overset{\text{em}}{T}_{\alpha\beta}$ for the electromagnetic field with zero trace.

5. Sources for the electromagnetic field—conserved current

$$\underset{\sim}{j}^{\alpha}=\frac{1}{2\pi}\,\mathbf{g}^{[\alpha\beta]}\partial_{\beta}(\mathbf{g}^{[\mu\nu]}F_{\mu\nu})-\frac{3}{4\pi}\,\partial_{\beta}\Psi(\mathbf{H}^{\beta\alpha}-2\mathbf{g}^{[\beta\alpha]}\cdot(\mathbf{g}^{[\mu\nu]}F_{\mu\nu}))$$

6. An additional term in the equation of motion for a charged particle

$$-\frac{1}{8}\left(\frac{g}{m_0}\right)^2\tilde{\mathbf{g}}^{(\beta\alpha)}\left(\frac{1}{\rho^2}\right)_{,\beta}$$

or in terms of Ψ

$$-\frac{1}{4}\left(\frac{q}{m_0}\right)^2\tilde{\mathbf{g}}^{(\beta\alpha)}e^{2\Psi}\Psi_{,\beta}$$

7. Propagation of the scalar uncharged field Ψ (or ρ) [equation (8.7)] and a Lagrangian for Ψ [see equation (7.8)] with an interaction term involving the electromagnetic field

$$8\pi e^{-3\Psi}\mathscr{L}_{\text{em}}$$

which plays a role similar to a mass term.
8. An energy-momentum tensor for a scalar field with nonzero trace, which suggests that this field is massive.
9. An interpretation of the scalar field as a gravitational "constant"

$$K=e^{-3\Psi}$$

10. Points 7–9 suggest that the scalar force is of short range. Thus, it does not violate the equivalence principle. It allows the gravitational "constant" to be really constant at long distances, and the additional component in the equation of motion for a charged test particle (see point 6) goes to zero.

All of these effects vanish if the skew-symmetric part of the metric is zero. We then get the classical results.

Let us write in this section a general form of the Lagrangian, coming back to CGS units. We get

$$\mathscr{L}=R(\bar{W})+8\pi(\mu\lambda)^2e^{-3\Psi}\mathscr{L}_{\text{em}}+\frac{8\pi G_N}{c^4}\left(\frac{1}{8\pi}\mathscr{L}_{\text{scal}}(\Psi)\right) \tag{8.27}$$

where

$$\lambda=\frac{2\sqrt{G_N}}{c^2}\quad\text{and}\quad\lambda\mu=\frac{2(G_N\hbar)^{1/2}}{c\hbar q}=\frac{2l_{\text{pl}}}{\sqrt{\alpha_{\text{em}}}} \tag{8.28}$$

where $l_{pl}=(G_N\hbar/C^3)^{1/2}$ is the Planck length and $\alpha_{em}=q^2/\hbar c$ is the fine structure constant.

Let us remark that we have three equivalent forms of the energy-momentum tensor for an electromagnetic field in our theory.

Let us write them down:

$$\overset{em(1)}{T}_{\alpha\beta}=\frac{1}{4\pi}\{g_{\gamma\beta}g^{\tau\rho}g^{\varepsilon\gamma}H_{\rho\alpha}H_{\tau\varepsilon}-2g^{[\mu\nu]}F_{\mu\nu}F_{\alpha\beta}$$
$$-\tfrac{1}{4}g_{\alpha\beta}[H^{\mu\nu}F_{\mu\nu}-2(g^{[\mu\nu]}F_{\mu\nu})^2]\} \tag{8.29}$$

$$\overset{em(2)}{T}_{\mu\nu}=\frac{1}{4\pi}\{g^{\alpha\beta}H_{\beta\nu}H_{\alpha\mu}-2g^{[\alpha\beta]}F_{\alpha\beta}F_{\mu\nu}$$
$$-\tfrac{1}{4}g_{\mu\nu}[H^{\alpha\beta}H_{\alpha\beta}-2(g^{[\alpha\beta]}F_{\alpha\beta})^2]\}-\frac{1}{8\pi}J_{\mu\nu} \tag{8.30}$$

$$J_{\mu\nu}=4H_{\alpha\mu}H_{\beta\nu}g^{[\alpha\beta]}-4H_{\alpha\mu}H_{\tau\varepsilon}g^{\tau\alpha}g_{\beta\nu}g^{[\varepsilon\beta]} \tag{8.31}$$

$$\overset{em(3)}{T}_{\alpha\beta}=\frac{1}{4\pi}\{g_{\sigma\beta}H^{\mu\sigma}F_{\mu\sigma}-2g^{\mu\nu}F_{\mu\nu}F_{\alpha\beta}$$
$$-\tfrac{1}{4}g_{\alpha\beta}[H^{\mu\nu}F_{\mu\nu}-2(g^{\mu\nu}F_{\mu\nu})^2]\} \tag{8.32}$$

It is easy to see that

$$g^{\alpha\beta}\overset{em(1)}{T}_{\alpha\beta}=g^{\alpha\beta}\overset{em(2)}{T}_{\alpha\beta}=g^{\alpha\beta}\overset{em(3)}{T}_{\alpha\beta}=0 \tag{8.33}$$

$T^{em(1)}_{\alpha\beta}$ has been considered in this section an energy-momentum tensor for the electromagnetic field $T^{em(2)}_{\alpha\beta}$ in Ref. 41 and $T^{em(3)}_{\alpha\beta}$ in Ref. 31. They are equivalent *modulo* equations (6.22)–(6.24). In Refs. 24 and 69 we consider $T^{em(1)}_{\alpha\beta}$ for a general and Abelian $[G=U(1)]$ gauge field.

Let us consider two 2-forms

$$\bar{H}=\pi^*(\tfrac{1}{2}H_{\mu\nu}\bar{\theta}^\mu\wedge\bar{\theta}^\nu)$$

and

$$\bar{M}=\pi^*(\tfrac{1}{2}M_{\mu\nu}\bar{\theta}^\mu\wedge\bar{\theta}^\nu)$$

One easily writes

$$\bar{H}=\Omega-4\pi\bar{M}=\Omega-\tfrac{1}{2}Q^5 \tag{8.34}$$

where $Q^5=\frac{1}{2}Q^5{}_{\mu\nu}\theta^\mu\wedge\theta^\nu$.

In this way we define the 2-form of induction $\bar{H}$ and find its geometrical interpretation in terms of the curvature and the torsion in the fifth dimension, as in Ref. 18. Equation (8.14a) can be rewritten in the following form using three-dimensional vectors and 3 × 3 matrices:

$$(aJ+A)\cdot \mathbf{D}+\mathbf{V}+\mathbf{H}=2A\mathbf{E}$$
$$(bJ+K)\cdot \mathbf{D}-\mathbf{W}\times\mathbf{H}=2b\mathbf{E}-2\mathbf{W}\times\mathbf{B} \tag{8.35}$$
$$(\mathbf{V}-\mathbf{Q})\cdot\mathbf{D}=2\mathbf{V}\mathbf{E}$$

and

$$(K*\bar{H}-A*\bar{H})+(\mathbf{W}-\mathbf{U})\otimes\mathbf{D}=2K*\bar{F}+2\mathbf{W}\otimes\mathbf{E} \tag{8.36}$$

where

$$a=(\mathbf{g}^{-1}\mathbf{g}^{T})_{44}, \qquad b=(\mathbf{g}^{T}\mathbf{g}^{-1})^{T}_{44} \tag{8.37}$$
$$\mathbf{V}=((\mathbf{g}^{T}\mathbf{g}^{-1})_{4\tilde{c}}), \qquad \tilde{c}=1,2,3$$
$$\mathbf{W}=((\mathbf{g}^{-1}\mathbf{g}^{T})_{4\tilde{c}}), \qquad \tilde{a}=1,2,3$$
$$\mathbf{U}=((\mathbf{g}^{T}\mathbf{g}^{-1})_{\tilde{b}4}), \qquad \tilde{b}=1,2,3 \tag{8.38}$$
$$\mathbf{Q}=((\mathbf{g}^{-1}\mathbf{g}^{T})_{\tilde{c}4}), \qquad \tilde{c}=1,2,3$$
$$A^{T}=((\mathbf{g}^{T}\mathbf{g}^{-1})_{\tilde{a}\tilde{c}}), \qquad \tilde{a}\tilde{c}=1,2,3 \tag{8.39}$$
$$K=((\mathbf{g}^{-1}\mathbf{g}^{T})_{\tilde{c}\tilde{b}}), \qquad \tilde{c},\tilde{b}=2,3,4$$

$*$ means matrix multiplication in three-dimensional space,
$\otimes$ means the tensor product of three-dimensional vectors, a dot
$(\cdot)$ means the scalar product in three-dimensional Euclidean space,
$A\mathbf{E}$ means the action of a 3 × 3 matrix on a three-dimensional vector,

$$\mathbf{E}=(E_{\tilde{a}})=(F_{\tilde{a}4}) \tag{8.40}$$
$$\mathbf{D}=(D_{\tilde{a}})=(H_{\tilde{a}4})$$
$$\bar{F}=(\bar{F}_{\tilde{m}\tilde{n}})=(\varepsilon_{\tilde{m}\tilde{n}\tilde{s}}B_{\tilde{s}})=-\bar{F}^{T} \tag{8.41}$$
$$\bar{H}=(\bar{H}_{\tilde{m}\tilde{n}})=(\varepsilon_{\tilde{m}\tilde{n}\tilde{s}}H_{\tilde{s}})=-\bar{H}^{T}$$

$\times$ is the vector product in a three-dimensional Euclidean space. Here $\varepsilon_{\tilde{m}\tilde{n}\tilde{s}}$ means the Levi–Civita symbol in three-dimensional Euclidean space. One easily gets that

$$\mathbf{B}=(B_{\tilde{s}})=(\tfrac{1}{2}\varepsilon_{\tilde{s}\tilde{m}\tilde{n}}\bar{F}_{\tilde{m}\tilde{n}}) \tag{8.42}$$

and

$$\mathbf{H}=(H_{\tilde{s}})=(\tfrac{1}{2}\varepsilon_{\tilde{s}\tilde{m}\tilde{n}}\bar{H}_{\tilde{m}\tilde{n}}) \tag{8.43}$$

In this way we lose the covariancy of equation (8.14), but we get the relations between three-dimensional vectors (**E**, **B**) and (**D**, **H**) as in an anisotropic dielectromagnetic medium.

For example, if we demand $\mathbf{D}=0$ $(\mathbf{E}\neq 0)$, we get

$$\mathbf{V}\times\mathbf{H}=2A\cdot\mathbf{E} \tag{8.35a}$$

$$\mathbf{W}\times\mathbf{H}=2\mathbf{W}\times\mathbf{B}-2b\mathbf{E}$$

$$\mathbf{V}\cdot\mathbf{E}=0$$

$$\tfrac{1}{2}(K*\bar{H}-A*\bar{H})=K*\bar{F}+\mathbf{W}\otimes\mathbf{E} \tag{8.36a}$$

Thus, $\mathbf{V}\perp\mathbf{E}$ and

$$H=2(K-A)^{-1}*(K*\bar{F}+\mathbf{W}\otimes\mathbf{E}) \tag{8.44}$$

if $\det(K-A)\neq 0$.

Using equation (8.35a), one gets

$$[(K-A)^{-1}*(K*\bar{F}+\mathbf{W}\otimes\mathbf{E})]\cdot\mathbf{V}=A\mathbf{E}$$

$$[(K-A)^{-1}*(K*F+\mathbf{W}\otimes\mathbf{E})]\cdot\mathbf{W}=2\mathbf{W}\times\mathbf{B}-2b\mathbf{E} \tag{8.45}$$

Thus, equation (8.45) with $\mathbf{V}\perp\mathbf{E}$ should be considered a condition for **g**, **E**, and **B** for a solution of the field equation with dielectric confinement, i.e., for $\mathbf{D}=0$ (no charge distribution in the presence of an electric field).

Moreover, in our theory there is a different tensor H, i.e.,

$$H^{\mu\alpha}=\mathbf{g}^{\beta\mu}\mathbf{g}^{\gamma\alpha}H_{\beta\gamma} \tag{8.46}$$

Thus, we can connect vectors **D** and **H** to this tensor, i.e.,

$$\begin{aligned}\mathbf{D}&=(D_{\bar{a}})=(H^{\bar{a}4}), \qquad \bar{a}=1,2,3\\ \mathbf{H}&=(H_{\bar{s}})=(\tfrac{1}{2}\varepsilon^{\bar{s}\bar{m}\bar{n}}\bar{H}_{\bar{m}\bar{n}}), \qquad \bar{s}=1,2,3\\ H&=(\bar{H}_{\bar{m}\bar{n}})=(\tfrac{1}{2}\varepsilon^{\bar{m}\bar{n}\bar{s}}H_{\bar{s}})\end{aligned} \tag{8.47}$$

In this case we should rewrite equation (8.14a) in terms of $H^{\mu\nu}$. We get

$$\mathbf{g}_{\mu\beta}\mathbf{g}_{\alpha\rho}H^{\mu\rho}+\mathbf{g}_{\alpha\delta}\mathbf{g}^{\delta\gamma}\mathbf{g}_{\gamma\rho}\mathbf{g}_{\beta\mu}H^{\mu\rho}=2\mathbf{g}_{\alpha\delta}\mathbf{g}^{\delta\gamma}F_{\beta\gamma} \tag{8.48}$$

We also find

$$\begin{aligned}&[2\mathbf{g}_{44}\mathbf{g}_{[\bar{m}4]}+\mathbf{g}_{4\delta}\mathbf{g}^{\delta\gamma}(\mathbf{g}_{\gamma4}\mathbf{g}_{1\bar{m}}-\mathbf{g}_{44}\mathbf{g}_{\gamma\bar{m}})]D_{\bar{m}}\\&\quad+\mathbf{g}_{4\bar{r}}(\mathbf{g}_{\bar{m}4}-\mathbf{g}_{4\delta}\mathbf{g}^{\delta\gamma}\mathbf{g}_{\gamma\bar{m}})\bar{H}_{\bar{m}\bar{r}}=-2\mathbf{g}_{4\delta}\mathbf{g}^{\delta\bar{c}}E_{\bar{c}}\end{aligned} \tag{8.49}$$

$$\begin{aligned}&[\mathbf{g}_{44}\mathbf{g}_{\bar{m}\bar{b}}-\mathbf{g}_{4\bar{b}}\mathbf{g}_{1\bar{m}}+\mathbf{g}_{4\delta}\mathbf{g}^{\delta\gamma}(\mathbf{g}_{\gamma4}\mathbf{g}_{\bar{b}\bar{m}}-\mathbf{g}_{\gamma\bar{m}}\mathbf{g}_{\bar{b}4})]D_{\bar{m}}\\&\quad+(\mathbf{g}_{\bar{m}\bar{b}}\mathbf{g}_{4\bar{r}}+\mathbf{g}_{4\delta}\mathbf{g}^{\delta\gamma}\mathbf{g}_{\gamma\bar{r}}\mathbf{g}_{\bar{b}\bar{m}})\bar{H}_{\bar{m}\bar{r}}=2\mathbf{g}_{4\delta}\mathbf{g}^{\delta\bar{c}}F_{\bar{b}\bar{c}}-2(\mathbf{g}_{4\delta}\mathbf{g}^{\delta4})E_{\bar{b}}\end{aligned} \tag{8.50}$$

$$[\mathbf{g}_{\bar{m}4}\mathbf{g}_{\bar{a}4}-\mathbf{g}_{44}(\mathbf{g}_{\bar{a}m}+\mathbf{g}_{\bar{a}\delta}\mathbf{g}^{\delta\gamma}\mathbf{g}_{\gamma m})+\mathbf{g}_{\bar{a}\delta}\mathbf{g}^{\delta\gamma}\mathbf{g}_{\gamma 4}\mathbf{g}_{4\bar{m}}]D_{\bar{m}}$$
$$+(\mathbf{g}_{\bar{m}4}\mathbf{g}_{\bar{a}\bar{r}}+\mathbf{g}_{4\bar{m}}\mathbf{g}_{\bar{a}\delta}\mathbf{g}^{\delta\gamma}\mathbf{g}_{\gamma\bar{r}})\bar{H}_{\bar{m}\bar{r}}=-2\mathbf{g}_{\bar{a}\delta}\mathbf{g}^{\delta\bar{c}}E_{\bar{c}} \tag{8.51}$$
$$(\mathbf{g}_{\bar{r}\bar{b}}\mathbf{g}_{\bar{a}4}-\mathbf{g}_{4\bar{b}}\mathbf{g}_{\bar{a}\bar{r}}+\mathbf{g}_{\bar{a}\delta}\mathbf{g}^{\delta\gamma}\mathbf{g}_{\gamma 4}\mathbf{g}_{\bar{b}\bar{r}}-\mathbf{g}_{\bar{a}\delta}\mathbf{g}^{\delta\gamma}\mathbf{g}_{\gamma\bar{r}}\mathbf{g}_{\bar{b}4})D_{\bar{r}}$$
$$+(\mathbf{g}_{\bar{m}\bar{b}}\mathbf{g}_{\bar{a}\bar{r}}+\mathbf{g}_{\bar{a}\delta}\mathbf{g}^{\delta\gamma}\mathbf{g}_{\gamma\bar{r}}\mathbf{g}_{\bar{b}\bar{m}})\bar{H}_{\bar{m}\bar{r}}=2\mathbf{g}_{\bar{a}\delta}\mathbf{g}^{\delta 4}E_{\bar{b}}+2\mathbf{g}_{a\delta}\mathbf{g}^{\delta\bar{c}}F_{\bar{b}\bar{c}} \tag{8.52}$$

Supposing $D_{\bar{r}}=0$ ($\mathbf{D}=0$), one gets

$$\mathbf{g}_{4\bar{r}}(\mathbf{g}_{\bar{m}4}-\mathbf{g}_{4\delta}\mathbf{g}^{\delta\gamma}\mathbf{g}_{\gamma\bar{m}})\bar{H}_{\bar{m}\bar{r}}=-2\mathbf{g}_{4\delta}\mathbf{g}^{\delta\bar{c}}E_{\bar{c}} \tag{8.53}$$
$$(\mathbf{g}_{\bar{m}\bar{b}}\mathbf{g}_{4\bar{r}}+\mathbf{g}_{4\delta}\mathbf{g}^{\delta\gamma}\mathbf{g}_{\gamma\bar{r}}\mathbf{g}_{\bar{b}\bar{m}})\bar{H}_{\bar{m}\bar{r}}=2\mathbf{g}_{4\delta}\mathbf{g}^{\delta\bar{c}}F_{\bar{b}\bar{c}}-2aE_{\bar{b}} \tag{8.54}$$
$$(\mathbf{g}_{\bar{m}\bar{4}}\mathbf{g}_{\bar{a}\bar{r}}+\mathbf{g}_{4\bar{m}}\mathbf{g}_{a\delta}\mathbf{g}^{\delta\gamma}\mathbf{g}_{\gamma\bar{r}})\bar{H}_{\bar{m}\bar{r}}=-2\mathbf{g}_{a\delta}\mathbf{g}^{\delta\bar{c}}E_{\bar{c}} \tag{8.55}$$
$$(\mathbf{g}_{\bar{m}\bar{b}}\mathbf{g}_{\bar{a}\bar{r}}+\mathbf{g}_{\bar{a}\delta}\mathbf{g}^{\delta\gamma}\mathbf{g}_{\gamma\bar{r}}\mathbf{g}_{\bar{b}\bar{m}})\bar{H}_{\bar{m}\bar{r}}=2\mathbf{g}_{\bar{a}\delta}\mathbf{g}^{\delta 4}E_{\bar{b}}+2\mathbf{g}_{\bar{a}\delta}\mathbf{g}^{\delta\bar{c}}F_{\bar{b}\bar{c}} \tag{8.56}$$

However, we should mention that a separation into space and time components of $\mathbf{g}_{\alpha\beta}$, $F_{\mu\nu}$, $H_{\mu\nu}$, and $H^{\mu\nu}$ is possible only if we deal with the stationary case. We suppose this in order to have a physical interpretation of the condition $\mathbf{D}=0$. Otherwise our considerations have a purely formal character.

A stationary space-time determines a three-dimensional manifold Σ_3 defined by the smooth map $\Phi: E \to \Sigma_3$, where $\Phi(x)$ denotes the trajectory of the timelike Killing vector $\bar{\eta}$. The elements of Σ_3 are orbits of the one-dimensional group of motions generated by $\bar{\eta}$. The 3-space Σ_3 is called the quotient space E/G_1. There is a one-to-one correspondence between tensor fields on Σ_3 and tensors on E, T satisfying $\bar{\eta}^\mu T_\mu{}^\nu=\bar{\eta}_\mu T_\nu{}^\mu=\mathscr{L}_{\bar{\eta}} T_\mu{}^\nu=0$, where $\bar{\eta}_\mu=\mathbf{g}_{(\mu\nu)}\bar{\eta}^\nu$. In our case we have on Σ_3 the following tensors:

$$h_{\mu\nu}=\mathbf{g}_{(\mu\nu)}+(-\mathbf{g}_{\alpha\beta}\,\bar{\eta}^\alpha\bar{\eta}^\beta)^{1/2}\bar{\eta}_\mu\bar{\eta}_\nu$$

and appropriate tensors built from $\mathbf{g}_{[\alpha\beta]}$, $F_{\mu\nu}$, $H_{\mu\nu}$, $H^{\mu\nu}$, etc. ($\eta=\partial/\partial x4$). The action of the group G_1 can be lifted to the electromagnetic bundle P and we get $\mathscr{L}_\eta\alpha=\mathscr{L}_\eta\gamma=0$, where $\eta=\Pi^*\bar{\eta}$. This corresponds to $\mathscr{L}_\eta\Omega=\mathscr{L}_\eta\bar{\omega}=0$ and consequently $\mathscr{L}_\eta\bar{H}=\mathscr{L}_\eta\bar{M}=0$. In the case of a static field configuration there is a natural way of introducing subspaces E_3 (orthogonal to the Killing trajectories).

Equations (8.52)–(8.55) should be considered consistency conditions of $D=0$. Thus, we can treat them as equations not only for $\bar{H}_{\bar{m}\bar{n}}$, but also for $\mathbf{g}_{44}$, $\mathbf{g}_{\bar{m}4}$, $\mathbf{g}_{4\bar{m}}$, and $\mathbf{g}_{\bar{m}\bar{n}}$ under the stationarity condition (the same condition for $\bar{H}_{\bar{m}\bar{n}}$, $E_{\bar{c}}$, $F_{\bar{b}\bar{c}}$). Thus, the dielectric confinement solution of the field equations can be derived from the second possibility, i.e., $\mathbf{D}=0$ obtained from $H^{\mu\nu}$.

9. EQUATION OF MOTION FOR A TEST PARTICLE. ADDITIONAL CONCLUSIONS

Let us come back to the (5.7) and consider it for $\rho=1$. Due to the compatibility condition (4.7), we have[70,71] the first integral of motion for equation (5.7),

$$\gamma(u(t), u(t)) = \gamma_{(AB)} u^A(t) u^B(t) = \text{const} \tag{9.1}$$

or

$$g_{(\alpha\beta)} u^\alpha(t) u^\beta(t) - (u^5)^2 = \text{const} \tag{9.1a}$$

However, due to equation (5.6) we have

$$u^5 = \text{const}' \tag{9.2}$$

Thus, we get

$$\gamma(\text{hor}(u(t)), \text{hor}(u(t))) = g_{(\alpha\beta)} u^\alpha(t) u^\beta(t) = \text{const}'' \tag{9.3}$$

We suppose const″ ≥ 0, i.e., we do not consider spacelike world-lines on E.

Let us rewrite equation (5.7) for $\rho=1$ in the following form:

$$m_0 a^\alpha + q g^{\alpha\gamma} F_{\gamma\beta} \left(\frac{dx^\beta}{dt}\right) - q g^{[\alpha\gamma]} H_{\gamma\beta} \left(\frac{dx^\beta}{dt}\right) = 0 \tag{9.4}$$

where

$$\frac{q}{m_0} = 2u^5 \tag{9.4a}$$

and

$$a^\alpha = \frac{\bar{D}u^\alpha}{dt} = \frac{\bar{D}}{dt}\left(\frac{dx^\alpha}{dt}\right) \tag{9.5}$$

is the covariant four-acceleration of a test particle. Equations (9.4) and (9.4a) are defined on an electromagnetic bundle P. Moreover, we can get them on E by taking any local section of P. For $F_{\mu\nu}$, $H_{\mu\nu}$, and ρ well defined on E (not dependent on a section), we get the same equations. Let us consider an initial Cauchy problem for (9.4) such that

$$\begin{gathered} x^\alpha(t_0) = x_0^\alpha \\ \frac{dx^\alpha}{dt}(t_0) = u_0^\alpha \\ g_{\alpha\beta} u_0^\alpha u_0^\beta = 1 \end{gathered} \tag{9.6}$$

i.e., we consider timelike curves on E. They have a natural interpretation as world-lines of massive test particles ($m_0 \neq 0$). In the case of null world-lines one has $g_{(\alpha\beta)} u_0^\alpha u_0^\beta = 0$ ($m_0 = 0$) and u^5 does not have a meaning as (q/m_0).

Due to equation (9.3) we have for every $t \geq t_0$

$$g_{\alpha\beta} \frac{dx^\alpha}{dt}(t) \frac{dx^\beta}{dt}(t) = 1 \tag{9.7}$$

Now we will find an interpretation of the additional term for the Lorentz force in equation (9.4), i.e.,

$$-q g^{[\alpha\gamma]} H_{\gamma\beta} \frac{dx^\beta}{dt} \tag{9.8}$$

To do this, let us consider equation (9.4) without this term, i.e.,

$$m_0 a^\alpha + q g^{\alpha\gamma} F_{\gamma\beta} \frac{dx^\beta}{dt} = 0 \tag{9.9}$$

This equation is a simple generalization of the equation for a charged point particle in general relativity to the nonsymmetric case. Now $g^{\alpha\gamma}$ is not symmetric and the covariant four-acceleration is defined in terms of the connection $\bar{\omega}^\alpha{}_\beta$ on E. This connection is of course compatible with the nonsymmetric metric $g_{\alpha\beta}$. One easily checks that

$$\frac{d}{dt}\left(g_{(\alpha\beta)} \frac{dx^\alpha}{dt} \frac{dx^\beta}{dt}\right) = -2u^5 g_{\alpha\delta} g^{\delta\gamma} F_{\gamma\beta} \left(\frac{dx^\beta}{dt}\right)\left(\frac{dx^\alpha}{dt}\right) \neq 0 \tag{9.10}$$

Thus, in general equation (9.9) does not have the first integral of motion (9.3). This means that we are unable in general to preserve the initial normalization for the four-velocity of a test particle. If we want to have the normalization (9.7), we must add to equation (9.9) the auxiliary condition

$$\Phi(U^\alpha) = g_{(\alpha\beta)} u^\alpha u^\beta - 1 = 0 \tag{9.11}$$

The auxiliary condition (9.11) is a nonholonomic constraint. This constraint is nonintegrable and nonlinear (quadratic in velocities). According to the general theory of mechanical systems with constraints, we know that in such systems we have the so-called reaction forces of constraints. Thus, we should write (9.9) in the following form:

$$m_0 a^\alpha = -(2u^5 m_0) g^{\alpha\gamma} F_{\gamma\rho} u^\rho + Q^\alpha \tag{9.12}$$

$$\Phi(u^\alpha) = g_{\alpha\beta} u^\alpha u^\beta - 1 = 0 \tag{9.13}$$

Q^α is a reaction force of the constraint (9.13). The force Q^α must be such that (9.13) is automatically satisfied during a motion. Let us find this force.

In order to do this, we multiply both sides of (9.12) by $g_{(\alpha\beta)}u^\beta$ and integrate from t_0 to t. We get

$$\frac{m_0}{2}\Phi(u^\alpha)=\frac{m_0}{2}(g_{(\alpha\beta)}u^\alpha u^\beta-1)$$
$$=\int_{t_0}^{t}(g_{(\alpha\beta)}u^\beta Q^\alpha-2m_0u^5g_{(\alpha\beta)}g^{\alpha\gamma}F_{\gamma\rho}u^\beta u^\rho)\,dt \tag{9.14}$$

If (9.13) is satisfied, we get

$$\int_{t_0}^{t}(g_{(\alpha\beta)}u^\beta Q^\alpha-2m_0u^5g_{(\alpha\beta)}F_{\gamma\rho}g^{\alpha\gamma}u^\beta u^\rho)\,dt=0 \tag{9.15}$$

Moreover, equation (9.15) is satisfied for every t. Thus, we get

$$g_{(\alpha\beta)}u^\beta Q^\alpha-2m_0u^5g_{(\alpha\beta)}g^{\alpha\gamma}F_{\gamma\rho}u^\beta u^\rho=0 \tag{9.16}$$

It is easy to see that equation (9.16) has a solution

$$Q^\alpha=2m_0u^5g^{\alpha\gamma}F_{\gamma\rho}u^\rho \tag{9.17}$$

If we put (9.17) into (9.12), we get

$$m_0a^\alpha=0 \tag{9.18}$$

This solution has simple physical interpretation. Equation (9.18) is an equation of motion for an uncharged test particle. There is no Lorentz force. It corresponds to a choice $u^5=0$ or equivalently $q=0$. Let us come back to equation (9.16) and transform it using the condition (4.9). We get

$$(g_{\alpha\beta}u^\beta Q^\alpha+g_{\beta\alpha}u^\beta Q^\alpha+m_0u^5(g_{\delta\beta}g^{\gamma\delta}H_{\gamma\alpha}+g^{\delta\gamma}H_{\beta\gamma})u^\alpha u^\beta)=0 \tag{9.19}$$

Equation (9.19) has a solution

$$Q^\alpha=2m_0u^5g^{[\alpha\gamma]}H_{\gamma\beta}u^\beta=qg^{[\alpha\gamma]}H_{\gamma\beta}u^\beta \tag{9.20}$$

Equation (9.19) gives us an interpretation for an additional term for the Lorentz force in equation (9.4). This additional term is a reaction force of the nonintegrable, nonholonomic, nonlinear constraints (9.11).

It is easy to see that our constraints are nonideal, for Q^α is not proportional to a gradient of Φ. The constraints seem to be similar to the so-called servo-constraints.

Let us consider a null world-line, i.e.,

$$g_{(\alpha\beta)}u^\alpha u^\beta=0 \tag{9.21}$$

In this case $m_0=0$ and we have $q=0$. Moreover, u^5 could be nonzero. For $u^5=0$ we get $a^\alpha=0$, i.e., the usual photon trajectory in NGT. If $u^5\neq 0$, we get the equation of a "charged photon" where u^5 is a measure of its coupling to the electromagnetic field.

Let us pass to the field $H_{\alpha\beta}$. This field plays the role of the second tensor of the electromagnetic strength. However, we have to do with only one electromagnetic field. Equation (4.9) expresses the relationship between $F_{\alpha\beta}$ and $H_{\alpha\beta}$. This is a linear equation for $H_{\alpha\beta}$. The difference between $H_{\alpha\beta}$ and $F_{\alpha\beta}$ appears due to the skew-symmetric part of the metric $g_{\alpha\beta}$. If $g_{[\alpha\beta]}=0$, we have $H_{\alpha\beta}=F_{\alpha\beta}$. The second pair of Maxwell equations (8.6) is the same as in nonlinear electrodynamics or in the classical electrodynamics of continuous media. In (8.6) we have a source, a conserved current. This current depends on the skew-symmetric part of the metric. In the nonsymmetric theory of gravitation the fermion current is the source for the differential equation for $g_{[\mu\nu]}$. In this way the fermion current becomes the source of the difference between $H_{\alpha\beta}$ and $F_{\alpha\beta}$. In the nonsymmetric theory of gravitation there is no Lorentz-like force term connected with a fermion charge (see Refs. 34, 63, 64).

This is a very important property of this theory. Due to this, the weak equivalence principle is satisfied, i.e., the universal falling of all uncharged bodies. This statement is not true for charged bodies. We have the Lorentz force term. In the nonsymmetric Kaluza–Klein theory there appears an additional term involving the tensor $H_{\alpha\beta}$ and the skew-symmetric part of the metric $g^{[\alpha\gamma]}$. Due to this term, the fermion charge has an influence on the motion of the test particle. It is of course an influence via a gravitational and an electromagnetic field (no additional Lorentz force with the fermion charge of a particle). But it is an influence. For example, the exact static, spherically symmetric solution of Moffat's theory has two sources: a mass point m and a point fermion charge l^2.[34,63,64]

Let us pass to equation (4.9). We are able to solve this equation by using iterative methods for the weak gravitational field. In order to do this, we write (4.9) in the form

$$H_{\beta\alpha}=(g_{\alpha\delta}g^{\delta\gamma}F_{\beta\gamma}-g_{[\delta\beta]}g^{\gamma\delta}H_{\gamma\alpha}-g_{[\alpha\delta]}g^{\delta\gamma}H_{\beta\gamma}) \tag{9.22}$$

and define the following transformation:

$$\overset{(n+1)}{H}_{\beta\alpha}=M^{\mu\nu}{}_{\beta\alpha}\overset{(n)}{H}_{\mu n} \tag{9.23}$$

such that

$$\overset{(0)}{H}_{\beta\alpha}=F_{\beta\alpha} \tag{9.24}$$

$$\overset{(n+1)}{H}_{\beta\alpha}=(g_{\alpha\delta}g^{\delta\gamma}F_{\beta\gamma}-g_{[\delta\beta]}g^{\gamma\delta}\overset{(n)}{H}_{\gamma\alpha}-g_{[\alpha\delta]}g^{\delta\gamma}\overset{(n)}{H}_{\beta\gamma}),\qquad n=0,1,2,\ldots \tag{9.24a}$$

One easily gets that

$$\overset{(n+1)}{H}_{\beta\alpha}=(M^{n+1})^{\mu\nu}{}_{\beta\alpha}\overset{(0)}{H}_{\mu\nu}=(M^{n+1})^{\mu\nu}{}_{\beta\alpha}F_{\mu\nu} \tag{9.25}$$

The index $(n+1)$ means the $(n+1)$ iteration of the transformation (9.23). We get

$$\overset{(n+1)}{H}_{\beta\alpha}-\overset{(n)}{H}_{\beta\alpha}=-[\mathbf{g}_{[\delta\beta]}\mathbf{g}^{\gamma\delta}(\overset{(n)}{H}_{\gamma\alpha}-\overset{(n-1)}{H}_{\gamma\alpha})+\mathbf{g}_{[\alpha\delta]}\mathbf{g}^{\delta\gamma}(\overset{(n)}{H}_{\beta\gamma}\overset{(n-1)}{H}_{\beta\gamma})] \tag{9.26}$$

Now let us suppose that the field $\mathbf{g}_{\alpha\beta}$ is weak. This means that

$$\mathbf{g}_{\alpha\beta}=\eta_{\alpha\beta}+h_{\alpha\beta} \tag{9.27a}$$

$$\mathbf{g}^{\alpha\beta}=\eta^{\alpha\beta}+\tilde{h}^{\alpha\beta} \tag{9.27b}$$

$$|h_{\alpha\beta}|,|\tilde{h}^{\alpha\beta}|<\alpha\ll 1 \tag{9.28}$$

where $\eta_{\alpha\beta}$ is the Minkowski tensor. In this case one gets

$$\mathbf{g}^{\mu\delta}\cong\eta^{\mu\delta}-\eta^{\alpha\mu}\eta^{\gamma\delta}h_{\gamma\alpha} \tag{9.29}$$

The skew-symmetric tensors

$$L_{\beta\nu}=-L_{\nu\beta} \tag{9.30}$$

form a natural linear six-dimensional vector space. Let us define the following norm in this space:

$$\|L\|=\max_{\beta,\nu=1,2,3,4}|L_{\beta\nu}| \tag{9.31}$$

Thus, our space becomes a Banach space. For sufficiently small α one finds

$$\|\overset{(n+1)}{H}-\overset{(n)}{H}\|\leq\beta(\alpha)\|\overset{(n)}{H}-\overset{(n+1)}{H}\| \tag{9.32}$$

where $0<\beta(\alpha)=96\alpha<1$; if $0<\alpha<1/96$, equation (9.32) means that the transformation (9.23) is a contraction. According to the Banach theorem, this transformation has a fix point

$$\overset{(\infty)}{H}_{\beta\alpha}=M^{\mu\nu}{}_{\beta\alpha}\overset{(\infty)}{H}_{\mu\nu} \tag{9.33}$$

such that

$$\overset{(\infty)}{H}_{\beta\alpha}=\lim_{n\to\infty}\overset{(n)}{H}_{\beta\alpha}=\lim_{n\to\infty}(M^n)^{\mu\nu}{}_{\beta\alpha}F_{\mu\nu}=\overset{(\infty)}{M}{}^{\mu\nu}{}_{\alpha\beta}F_{\mu\nu} \tag{9.34}$$

The limit (9.34) is understood in the sense of the norm (9.31) and

$$\overset{(\infty)}{M}{}^{\mu\nu}{}_{\beta\alpha}=\lim_{n\to\infty}(M^n)^{\mu\nu}{}_{\beta\alpha} \tag{9.35}$$

The limit (9.35) is understood in the sense of the usual linear operator topology generated by the topology of a Banach space. According to the Banach theorem, there is one and only one fix point of the transformation (9.23) (in the weak-field approximation). Thus, we get that

$$H_{\beta\alpha}=\overset{(\infty)}{M}{}^{\mu\nu}{}_{\beta\alpha}F_{\mu\nu} \tag{9.36}$$

Equation (9.36) is the solution of equation (4.9). In this case the additional term for the Lorentz force in equation (9.4) takes the form

$$-q\mathbf{g}^{[\alpha\gamma]}\overset{(\infty)}{M}{}^{\mu\nu}{}_{\gamma\beta}u^{\beta}F_{\mu\nu} \tag{9.37}$$

It is purely described by the tensor $F_{\mu\nu}$ and the metric tensor $\mathbf{g}_{\alpha\beta}$. We have the same for the reaction force of constraints

$$Q^{\alpha}=-q\mathbf{g}^{[\alpha\gamma]}\overset{(\infty)}{M}{}^{\mu\nu}{}_{\gamma\beta}u^{\beta}F_{\mu\nu} \tag{9.38}$$

For nonholonomic (nonintegrable) constraints we have the following statement. A variational problem with differential (nonintegrable, nonholonomic) constraints cannot be reduced to a form where the variation of a certain quantity (an action) is put equal to zero. This is true in the much simpler case of linear nonholonomic constraints.[72] Thus, unfortunately, we cannot formulate a principle of action for equation (9.4). Moreover, we are still able to interpret the additional term in the Lorentz force as a reaction force of the nonholonomic constraints (9.11). However, we can try to formulate a local Gausslike principle in order to derive equation (9.4). Let us consider a local Gausslike principle for equation (9.4) in the following form:

$$\delta Z^2=0$$

modulo constraints (9.11), where

$$Z^2=\frac{m_0}{2}\tilde{\mathbf{g}}^{(\alpha\beta)}f_{\alpha\gamma}\left(a^{\gamma}-\frac{F^{\gamma}}{m_0}\right)f_{\beta\mu}\left(a^{\mu}-\frac{F^{\mu}}{m_0}\right)$$

$f_{\alpha\gamma}$ is defined as follows: $f_{\rho}{}^{\zeta}f^{\rho}{}_{\delta}=\mathbf{g}^{[\zeta\mu]}H_{\mu\delta}=h^{\zeta}{}_{\delta}$, $f^{\rho}{}_{\delta}=f^{\rho\nu}\mathbf{g}_{(\nu\delta)}$, and $f^{\rho\nu}f_{\rho\mu}=\delta^{\nu}{}_{\mu}$, $\det(F_{\rho\mu})\neq 0$.

Thus, f exists if the matrix $h^{\zeta}{}_{\delta}$ is invertible, symmetric, and positive definite. It seems that only in this case can we formulate a Gausslike principle for equation (9.4).

Thus, we get

$$\tilde{\mathbf{g}}^{(\alpha\beta)}f_{\alpha\gamma}(m_0a^{\gamma}-F^{\gamma})f_{\beta\nu}\delta a^{\nu}=0$$

We recall that for the Gauss principle we are taking the variation with respect to the accelerations only. The acceleration a^{α} is a covariant acceleration with respect to the connection $\bar{\Gamma}^{\alpha}{}_{\beta\gamma}$ on E.

Differentiating equation (9.11) with respect to t, one gets

$$0=\frac{d}{dt}\Phi(u^{\alpha})=\frac{\bar{D}}{dt}\Phi(u^{\alpha})=\frac{\partial\Phi}{\partial u^{\alpha}}\frac{\bar{D}u^{\alpha}}{dt}+\frac{\partial\Phi}{\partial g_{(\alpha\beta)}}\frac{\bar{D}}{dt}g_{(\alpha\beta)}$$

$$=\frac{\partial\Phi}{\partial u^{\alpha}}a^{\alpha}+D_2\Phi=2g_{(\alpha\beta)}u^{\beta}a^{\alpha}+D_2\Phi$$

Thus, the allowed variations of the accelerations satisfy the condition

$$\frac{\partial\Phi}{\partial u^{\alpha}}\delta a^{\alpha}=2g_{(\alpha\beta)}u^{\beta}\delta a^{\alpha}=0$$

From the above equations we get

$$\tilde{g}^{(\alpha\beta)}f_{\alpha\gamma}(m_0a^{\gamma}-F^{\gamma})f_{\beta\nu}+2rg_{(\nu\beta)}u^{\beta}=0$$

where r is a Lagrange multiplier. Using the definition of $f_{\alpha\beta}$, we come to equation (9.4) and $r=-q/2$. The force Q^{α} can be expressed in terms of the ideal reaction force R_{α}, i.e.,

$$Q^{\alpha}=\mu P^{\alpha}{}_{\nu}R_{\alpha}, \qquad \mu\neq 0$$

where

$$P^{\alpha}{}_{\nu}=g^{[\alpha\gamma]}H_{\gamma\nu}$$

Note that the conditions for the application of a Gausslike principle are as follows:

1. $\det[(g_{\alpha\zeta}g^{\zeta\mu}-g_{\zeta\alpha}g^{\mu\zeta})H_{\mu\delta}]\neq 0$.
2. The matrix $h_{\alpha\delta}=(g_{\alpha\zeta}g^{\zeta\mu}-g_{\zeta\alpha}g^{\mu\zeta})H_{\mu\delta}$ is positively defined and symmetric. Let us note the following facts. We formulate a local Gausslike principle for the equation of motion for a test particle in nonsymmetric Kaluza–Klein theory (NKKT). Moreover, the original equation has been derived from a Galilei like principle in NKKT. According to this principle, test particles move along the simplest lines in NKKT (an extended Galilei-like principle states this). Moreover, we can get the equation in a different way, formulating a local Gausslike principle. This principle is generally covariant and the acceleration a^{α} is a covariant four-acceleration with respect to the nonsymmetric connection on E. The constraints are also covariant and depend explicitly on t—a parameter along the particle trajectory. In this way we get the interpretation of an additional term for a Lorentzlike force as a nonideal reaction force (it is not proportional to the gradient to the hypersurface of constraints). The application of a Gausslike principle is

possible under some assumptions concerning $g_{\alpha\beta}$ and $F_{\mu\nu}$. Moreover, they can be satisfied and we get a proper equation (i.e., those obtained from a Galileilike principle in NKKT). The above Gausslike principle can be considered a minimum (extremum) principle for a quadratic function of accelerations *modulo* constraints. For F^α we have

$$F^\alpha = qg^{\alpha\beta}F_{\beta\gamma}u^\gamma$$

During the motion Z^2 is minimalized (extremalized) *modulo* the nonlinear nonholonomic constraints (9.7). The constraints are nonideal and the force Q^α is a nonideal reaction force.

From the geometrical point of view (the force Q^α is absorbed by the geometry) it seems that only the metric geometry or the Einstein geometry defined on the five-dimensional Kaluza–Klein manifold lead to the condition (9.1). The geometry defined by the metric $\bar{g} = g_{(\alpha\beta)}\bar{\theta}^\alpha \otimes \bar{\theta}^\beta$, the 2-form $\underline{g} = g_{[\alpha\beta]}\bar{\theta}^\alpha \wedge \bar{\theta}^\beta$, and the connection $\bar{\omega}^\alpha{}_\beta$ satisfying the condition (4.7) we call the Einstein geometry. If we want to get conditions (9.2) and (9.3), it seems that we have only three possibilities:

1. Riemannian geometry (classical Kaluza–Klein theory).
2. A generalization of the Einstein–Cartan theory and the Kaluza–Klein theory.[16,69]
3. Einstein geometry on the electromagnetic bundle manifold, i.e., the theory described in Section 3.

The first two geometries are metric. The first one is only a model of a unification of electromagnetic and gravitational fields. This unification is too perfect. We do not get any "interference effects" between gravitational and electromagnetic fields. It seems that it is only a five-dimensional representation of general relativity and Maxwell's theory in Riemannian space-time. The second possibility, due to the Cartan equations on the space-time and in the fifth dimension, offers some interference effects: an additional current connected to spin sources. W. Israel's energy-momentum tensor as the tensor of an energy-momentum for the electromagnetic field, and a contact interaction term of electromagnetic polarization in the total energy-momentum tensor. Unfortunately, an additional geometric degree of freedom, torsion, is connected algebraically with external sources, spin and the electromagnetic polarization of matter. Thus, this torsion does not propagate. The third possibility seems to be more interesting. There are "interference effects" between gravitational and electromagnetic fields. Torsion propagates. It is interesting to notice that despite completely different geometries in the second and third possibilities, we get the same equation connecting the electromagnetic polarization existing in the theory to a torsion in the fifth dimension.

Let us pass finally to the following conclusion. The nonsymmetric Kaluza–Klein theory offers us a unified theory of the gravitational and electromagnetic fields. In this theory Einstein's unified field theory (real version) is treated as a theory of the pure gravitational field, according to Moffat's approach.[34,63,64] However, we can still have the old interpretation of Einstein's unified theory if we follow Klotz.[73] Due to the metric hypothesis, he interprets Einstein's theory (weak system of field equations) as a unified field theory of macroscopic gravitational and electromagnetic fields. The metric hypothesis means that

$$\tilde{\Gamma}^{\alpha}{}_{(\beta\gamma)}=\left\{\begin{matrix}\alpha\\ \beta\gamma\end{matrix}\right\}_{p} \tag{9.39}$$

where p is a metric tensor, which in general has nothing to do with $g_{(\alpha\beta)}$. Using equation (9.39), Klotz is able to get a Coulomb solution and Lorentz force term, which was impossible to get in previous approaches. He interprets $R_{[\mu\nu]}(\tilde{\Gamma})$ as $F_{\mu\nu}$—the strength of the electromagnetic field. In the linear approximation this is coherent with the previous interpretation,

$$F_{\mu\nu}\sim g^{\alpha\beta}g_{[\dot{\mu}\dot{\nu}];\underset{(+)}{\alpha\beta}} \tag{9.40}$$

where the dot means a covariant derivative with respect to $\tilde{\Gamma}^{\alpha}{}_{(\beta\gamma)}$. However, the condition (9.39) seems to be very restrictive and some solutions of Einstein's weak system of field equations do not satisfy equation (9.39).

The pure gravitational interpretation proposed by Moffat seems to be more fundamental. The nonsymmetric Kaluza–Klein theory offers a possible reinterpretation of NGT. According to equation (4.9), the tensor $H_{\alpha\beta}$ is expressible by $F_{\alpha\beta}$ and $g_{\alpha\beta}$. The equation is linear with respect to $H_{\alpha\beta}$ and can be solved. In Section 8 we define the tensor $M_{\alpha\beta}$. This tensor is skew-symmetric and if $g_{[\alpha\beta]}=0$, $M_{\alpha\beta}$ is zero, too. $M_{\alpha\beta}$ has the physical interpretation as the polarization tensor. Simultaneously we get the geometrical interpretation of $M_{\alpha\beta}$ as the torsion in the fifth dimension ($Q^{5}_{\alpha\beta}=8\pi M_{\alpha\beta}$). Thus, we come to the conclusion that it would be possible to reinterpret the nonsymmetric theory of gravitation as a theory with nonzero torsion in the fifth dimension as a fundamental quantity. In this way one rewrites equation (4.9)

$$g_{\delta\beta}g^{\gamma\delta}M_{\gamma\alpha}+g_{\alpha\delta}g^{\delta\gamma}M_{\beta\gamma}=\frac{1}{4\pi}(g_{\delta\beta}g^{\gamma\delta}F_{\gamma\alpha}-g_{\alpha\delta}g^{\delta\gamma}F_{\beta\gamma}) \tag{9.41}$$

Using matrix notation, we can rewrite equation (9.41) in the following form:

$$g(g^{-1})^T M + g^T g^{-1} M^T = \frac{1}{4\pi}[g(g^{-1})^T F - g^T g^{-1} F^T] \tag{9.41a}$$

where T means matrix transposition.

We can treat equation (9.41) as an algebraic equation for $\mathbf{g}_{[\alpha\beta]}$ and $M_{\alpha\beta}$ as a known quantity. We have the same number of degrees of freedom for $M_{\alpha\beta}$ and $\mathbf{g}_{[\alpha\beta]}$. Equation (9.41) is nonlinear with respect to $\mathbf{g}_{[\alpha\beta]}$ and more difficult to solve. In this way we can reinterpret the full theory as a theory with torsion in the fifth dimension. Thus, our theory has many similarities with previous approaches, i.e., the Kaluza–Klein theory with torsion.(16,68)

Let us consider equation (9.41) in more detail, trying to solve it using iterative methods (i.e., generalized Newton–Kantorowicz method). In order to do this, we consider a 16-dimensional Banach space of 4×4 matrices $(\mathbf{g}_{\alpha\beta}) = g$ with a natural norm of operators induced by a Euclidean norm in four-dimensional Banach space. Let us denote it by $\mathscr{X} = (X, \|\circ\|)$. We define a nonlinear operator acting in X for such $g = (\mathbf{g}_{\alpha\beta})$ that $\det(\mathbf{g}_{\alpha\beta}) \neq 0$,

$$T:\quad X \to X \tag{9.42}$$

We have $D(T) = \{\mathbf{g}_{\alpha\beta}, \det(\mathbf{g}_{\alpha\beta}) \neq 0\}$ and it is open in $\mathscr{X}$,

$$T(\mathbf{g}_{\rho\psi}))_{\alpha\beta} = \mathbf{g}_{\delta\beta} g^{\gamma\delta}\left(M_{\gamma\alpha} - \frac{1}{4\pi} F_{\gamma\alpha}\right) + \mathbf{g}_{\alpha\delta} g^{\delta\gamma}\left(M_{\beta\gamma} + \frac{1}{4\pi} F_{\beta\gamma}\right) \tag{9.43}$$

One easily notices that $T(\kappa g) = T(g)$ for $g \in D(T)$, $\kappa \neq 0$. Thus, we can consider an equivalence $g_1 \sim g_2$ if $g_1 = \kappa g_2$.

Let us denote an equivalence class of g by $[g]$. One easily notices that there is $\tilde{g} \in [g]$ such that $\|\tilde{g}\| = 1$. Thus, we can consider T in $D(T) \cap S(1)$, where $S(1) = \{\mathbf{g}_{\alpha\beta},\ \|g\| = 1\}$.

The solution of equation (9.43) is given by

$$T(g_0) = 0 \tag{9.44}$$

It is easy to see that if $T(g_0) = 0$, then $T(\kappa g_0) = 0$ as well for $\kappa \neq 0$. Let us notice that the operator T is continuous in $\mathscr{X}$ and it possesses Frechet derivatives of any order at any point of $D(T)$. They are bounded linear (multilinear) operators in $\mathscr{X}$. Let us find the first and second derivatives of T at $g \in D(T) \subset X$. We get

$$\begin{aligned}((dT)|_g)^{\mu\nu}{}_{\alpha\beta} &= (\delta^\nu_\beta \mathbf{g}^{\gamma\mu} - \mathbf{g}_{\delta\beta}\mathbf{g}^{\gamma\nu}\mathbf{g}^{\mu\delta})\left(M_{\gamma\alpha} - \frac{1}{4\pi} F_{\gamma\alpha}\right) \\ &\quad + (\delta^\mu_\alpha \mathbf{g}^{\nu\gamma} - \mathbf{g}_{\alpha\delta}\mathbf{g}^{\delta\nu}\mathbf{g}^{\mu\gamma})\left(M_{\beta\gamma} + \frac{1}{4\pi} F_{\beta\gamma}\right)\end{aligned} \tag{9.45}$$

and

$$
\begin{aligned}
&((d^2T)|_g)^{\mu\nu,\rho\psi}{}_{\alpha\beta} \\
&= (\mathbf{g}_{\delta\beta}\mathbf{g}^{\gamma\psi}\mathbf{g}^{\rho\nu}\mathbf{g}^{\mu\delta} + \mathbf{g}_{\delta\beta}\mathbf{g}^{\gamma\nu}\mathbf{g}^{\mu\psi}\mathbf{g}^{\rho\delta} - \delta^{\nu}_{\beta}\mathbf{g}^{\gamma\psi}\mathbf{g}^{\rho\mu} - \delta^{\psi}_{\beta}\mathbf{g}^{\gamma\nu}\mathbf{g}^{\nu\rho}) \\
&\quad\times\left(M_{\gamma\alpha} - \frac{1}{4\pi}F_{\gamma\alpha}\right) \\
&\quad+ (\mathbf{g}_{\alpha\delta}\mathbf{g}^{\delta\psi}\mathbf{g}^{\rho\nu}\mathbf{g}^{\mu\gamma} + \mathbf{g}_{\alpha\delta}\mathbf{g}^{\delta\nu}\mathbf{g}^{\mu\psi}\mathbf{g}^{\rho\gamma} - \delta^{\mu}_{\alpha}\mathbf{g}^{\nu\psi}\mathbf{g}^{\rho\gamma} - \delta^{\rho}_{\alpha}\mathbf{g}^{\psi\nu}\mathbf{g}^{\mu\gamma}) \\
&\quad\times\left(M_{\beta\alpha} - \frac{1}{4\pi}F_{\beta\alpha}\right)
\end{aligned}
\tag{9.46}
$$

One easily finds

$$
|||(dT)|_g||| \le 2\|g^{-1}\|(1+\|g\|\,\|g^{-1}\|)\left(\|M\| + \frac{\|F\|}{4\pi}\right) \tag{9.47}
$$

and

$$
|||(d^2T)|_g||| \le 4\|g^{-1}\|^2(1+\|g\|\,\|g^{-1}\|)\left(\|M\| + \frac{\|F\|}{4\pi}\right) \tag{9.48}
$$

where $\|g^{-1}\|$ is the norm of the matrix $\mathbf{g}^{\alpha\beta}$, $\|M\|$ the norm of $M_{\alpha\beta}$, $\|F\|$ the norm of $F_{\alpha\beta}$, and $\|g\|$ the norm of $\mathbf{g}_{\alpha\beta}$. Here $|||\cdot|||$ is the operator norm induced by $\|\cdot\|$ in $\mathscr{X}$. One easily gets that $dT_{|\kappa g} = (1/\kappa)dT_{|g}$ and $d^2T_{|\kappa g} = (1/\kappa^2)d^2T_{|g}$.

$(d^2T)|_g$ is continuous (of course) with respect to g in $D(T)$. Let us consider $h_0 \in D(T)$ such that $(dT)|_{h_0}$ is invertible at h_0. This means that $((dT)|_{h_0})^{-1} = A_0 \in B(\chi, \chi)$ (it is bounded, of course). One easily gets

$$
\begin{aligned}
|||A||| &\le \frac{|||d(T)|_{h_0}|||^{15}}{|\det((d(T)|_{h_0})|} \\
&\le \frac{2^{15}\|h_0^{-1}\|^{15}(1+\|h_0\|\,\|h_0^{-1}\|)^{15}(\|M\| + \|F\|/4\pi)^{15}}{|\det((d(T)|_{h_0})|}
\end{aligned}
\tag{9.49}
$$

From (9.49) and (9.48) one finds

$$
\begin{aligned}
&|||A_0(d^2T)|_g||| \\
&\le 2^{17}\left(\|M\| + \frac{\|F\|}{4\pi}\right)^{16} \frac{\|g\|^2}{\|\det(g)\|}\left(1 + \frac{\|g\|^2}{\|\det(g)\|}\right)\|h^0\|^{45}\left(1 + \frac{\|h_0\|^4}{\|\det(h_0)\|}\right)^{15} \\
&\quad\times|\det((d(T)|_{h_0})|^{-1}\cdot|\det(h_0)|^{-15}
\end{aligned}
\tag{9.50}
$$

We use for $\|g^{-1}\|$ the following inequality:

$$\|g^{-1}\| \le \frac{\|g\|^3}{|\det(g)|} \tag{9.51*}$$

Let us calculate $\|A_0 T(h_0)\|$. We find

$$\|A_o T(h_0)\| \le |||A_0||| \cdot \|T(h_0)\|$$
$$\le \left(\frac{2(\|M\| + \|F\|/4\pi)}{|\det(h_0)|}\right)^{16} \frac{\|h_0\|^{49}(1 + \|h_0\|^4/\|\det(h_0)\|)^{15}}{|\det((d(T)|_{h_0})|} = \alpha \tag{9.51}$$

Let us consider a ball $\bar{K}(h_0, r)$, $r \le 1/\|h_0^{-1}\|$, $\bar{K}(h_0, r) \subset D(T)$, and $g \in \bar{K}(h_0, r)$. We define the function $d(r, h_0)$,

$$d(r, h_0) = \min_{g \in \bar{K}(h_0, r)} [|\det(g_{\alpha\beta})|] \tag{9.52}$$

We easily find for $g \in \bar{K}(h_0, r)$

$$|||A_0(d^2T)|_g|||$$
$$\le 2^{17}\left(\|M\| + \frac{\|F\|}{4\pi}\right)^{16} \|h_0\|^{45}\left(1 + \frac{\|h_0\|^4}{|h|}\right)^{15}$$
$$\times \left(\frac{\|h_0\|^3}{|h|} + \|h_0\|\right)^3 \left(1 + \frac{(\|h_0\|^3 + |h|\,\|h_0\|)^3}{|h|^4}\right)$$
$$\times \left[|h|^{15} d\left(\frac{\|h_0\|^3}{|h|}, h_0\right) \cdot D(h_0)\right]^{-1} = \beta \tag{9.53}$$

for $g \in \bar{K}(h, r)$, where $h = \det(h_0)$ and $D(h_0) = |\det(dT)|_{h_0}$.

Let us find the product $\alpha \cdot \beta$. We get

$$\alpha\beta = 2\left(\frac{2(\|M\| + \|F\|/4\pi)}{|h|}\right)^{32}$$
$$\times \frac{\|h_0\|^{94}(1 + \|h_0\|^4/|h|)^{30}(\|h_0\|^3/|h| + \|h_0\|)^3[|h| + (\|h_0\|^3/|h| + \|h_0\|)^3]}{D^2(h_0)d(\|h_0\|^3|h|, h_0)} \tag{9.54}$$

Let us define the following sequence:

$$\overset{(0)}{g} = h_0 \tag{9.55a}$$

$$\overset{(n+1)}{g} = \overset{(n)}{g} - ((dT)|\overset{(n)}{g})^{-1} T(\overset{(n)}{g}) \tag{9.55b}$$

If this sequence converges, the limit

$$\overset{(\infty)}{g} = \lim_{n=\infty} \overset{(n)}{g}$$

satisfies equation (9.44), i.e.,

$$T(\overset{(\infty)}{g})=0 \tag{9.56}$$

The sufficient condition for the convergence of (9.55a)–(9.55b) is

$$\alpha\beta \le \tfrac{1}{2} \qquad \text{and} \qquad r \ge \frac{1-(1-2\alpha\beta)^{1/2}}{\beta}$$

In this case the sequence (9.55a)–(9.55b) converges to the solution of equation (9.44). One gets

$$\|\overset{(\infty)}{g} - \overset{(n)}{g}\| \le \frac{1}{\beta \cdot 2^n}(2\alpha\beta)^{2^n} \tag{9.57}$$

This is the Kantorowicz method. Thus, the method converges quickly. Using the method, we find

$$\overset{(0)}{\mathbf{g}}_{\mu\nu}(M_{\rho\psi}(x), F_{\alpha\beta}(x)) = \tilde{\mathbf{g}}_{\mu\nu}(x) \tag{9.58}$$

and this is a nonsymmetric metric induced by the electromagnetic field and the polarization tensor $M_{\rho\psi}(x)$ equal to the torsion in the fifth dimension. The most important fact is that we get a nonzero skewon field

$$H_{\mu\nu}(x) = \overset{(0)}{\mathbf{g}}_{[\mu\rho]}(M_{\rho\psi}(x), F_{\alpha\beta}(x)) \tag{9.59}$$

which can be substituted into the field equations for gravitational and electromagnetic fields together with the symmetric part of the metric

$$\mathbf{g}_{\mu\nu}(x) = \mathbf{g}_{(\mu\nu)}(x) + h_{\mu\nu}(x) \tag{9.60}$$

We can reconsider all the formulas presented here in a little different formalism, introducing ordinary vector and matrix notation in 16-dimensional linear space. This means that for any pair of four-dimensional indices we introduce one 16-dimensional index, i.e.,

$$a = (\mu - 1) + \nu \tag{9.61}$$

It is easy to check that (9.61) is unambiguous. One has

$$
\begin{aligned}
T_{\alpha\beta} &\leftrightarrow T_a \\
\mathbf{g}_{\alpha\beta} &\leftrightarrow X_a \\
\frac{DT_{\alpha\beta}}{\delta g_{\mu\nu}} &\leftrightarrow \frac{\delta T_a}{\delta X_b} = A^b{}_a
\end{aligned}
\tag{9.62}
$$

This formalism can help us in some practical calculations.

Finally, let us notice that we can consider a simpler method, i.e., a different sequence $g^{(n)'}$

$$\overset{(0)}{g'} = h_0 \tag{9.63a}$$

$$\overset{(n+1)}{g'} = \overset{(n)}{g'} - ((dT)|_{h_0})^{-1} T(\overset{(n)}{g'}) \tag{9.63b}$$

This sequence converges to the solution of equation (9.41)

$$T(\overset{(\infty)}{g'}) = 0$$

However, the convergence is slower.

Equation (9.41) can have more than one solution in $D(T) \subset X$.

We do not mean here solutions obtained from g_0 $[T(g_0) = 0]$ in a trivial way by multiplying by a nonzero factor $\kappa \neq 0$, i.e., κg_0. We mean here solutions which belong to different equivalence classes, i.e., g_0 and g_0, such that $[g_0] \neq [g_{0'}]$. Finally, let us notice that because of $T(\kappa g) = T(g)$, we can always consider a weak-field approximation for $\mathbf{g}_{\mu\nu}$ even if g is not small. This is because if $T(g_0) = 0$, then $T(\kappa g_0) = 0$ such that

$$\kappa g_0 = \eta + h \tag{9.64}$$

where η is a Minkowski tensor and $\|h\| \ll 1$.

10. MATERIAL SOURCES. PALATINI VARIATIONAL PRINCIPLE AND FIELD EQUATIONS

In this section we consider material sources in the nonsymmetric Kaluza–Klein theory, i.e., an energy-momentum tensor of external sources, a fermion current, an electric current, and a spin-density tensor of external sources. We will deal with the case $\rho = 1$.

We introduce material sources and find equations for gravitational and electromagnetic fields in the presence of matter with nonzero fermion current and nonzero electric current. We define a new geometrical degree of freedom (a generalized contortion tensor) in a similar way as in the Einstein–Cartan extension of Moffat's theory (Einstein–Cartan–Moffat theory[74]). Simultaneously we introduce spin sources. We find equations for the gravitational and electromagnetic fields and the Cartan equation in this case.

In Section 4 we introduced two connections on $\underline{P}$, $\omega^A{}_B$ and $W^A{}_B$ (we change for convenience the notations $\omega^A{}_B$, $\tilde{\omega}^A{}_B$ and $W^A{}_B$, $\tilde{W}^A{}_B$). Now the connection $\tilde{\omega}^A{}_B$ does not satisfy the compatibility condition for γ_{AB}, but a different connection $\tilde{\Lambda}^A{}_B = \tilde{\Lambda}^A{}_{BC}\theta^C$ satisfies this condition:

$$\tilde{D}\gamma_{A+B-} = \tilde{D}\gamma_{AB} - \gamma_{AD}\tilde{Q}^D{}_{BC}(\tilde{\Lambda})\theta^C = 0 \tag{10.1}$$

$$\xi_5 \tilde{\Lambda}^A{}_B = 0 \tag{10.2}$$

where $\tilde{D}$ is the exterior covariant derivative with respect to the connection $\tilde{\Lambda}^A{}_B$ and $\tilde{Q}^D{}_{BC}(\tilde{\Lambda})$ is the tensor of torsion for the connection $\tilde{\Lambda}^A{}_B$. One easily finds

$$\tilde{\Lambda}^A{}_B = \left[\begin{array}{c|c} \pi^*(\tilde{\Lambda}^\alpha{}_\beta) + g^{\gamma\alpha}H_{\gamma\beta}\theta^5 & H_{\beta\gamma}\theta^\gamma \\ \hline g^{\alpha\beta}(H_{\gamma\beta} + 2F_{\beta\gamma})\theta^\gamma & 0 \end{array}\right] \tag{10.3}$$

where we have for $\tilde{\Lambda}^\alpha{}_{\beta\gamma}$

$$g_{\mu\nu,\sigma} - g_{\rho\nu}\tilde{\Lambda}^\rho{}_{\mu\sigma} - g_{\mu\rho}\tilde{\Lambda}^\rho{}_{\sigma\nu} = 0 \tag{10.4}$$

For $\tilde{W}^A{}_B$ we have, in a similar manner as in Section 4,

$$\tilde{W}^A{}_B = \tilde{\omega}^A{}_B - \tfrac{3}{4}\delta^A_B \tilde{W} \tag{10.5}$$

$H_{\beta\gamma} = -H_{\gamma\beta}$ is a tensor on E and satisfies the following condition:

$$g_{\sigma\beta}g^{\gamma\delta}H_{\gamma\alpha} + g_{\alpha\delta}g^{\delta\gamma}H_{\beta\gamma} = 2g_{\alpha\delta}g^{\delta\gamma}F_{\beta\gamma} \tag{10.6}$$

For the connection $\tilde{\omega}^A{}_B$ we have the following:

$$\tilde{\omega}^A{}_B = \left[\begin{array}{c|c} \pi^*(\omega^\alpha{}_\beta) + g^{\gamma\alpha}H_{\gamma\beta}\theta^5 & H_{\beta\gamma}\theta^5 \\ \hline g^{\alpha\beta}(H_{\gamma\beta} + 2F_{\beta\gamma})\theta^\alpha & 0 \end{array}\right] \tag{10.7}$$

Thus, we have on $\underline{P}$ all five-dimensional analogues of the quantities from Moffat's theory of gravitation,[34,63,64] i.e., $\tilde{W}^A{}_B$, $\tilde{\omega}^A{}_B$, $\tilde{\Lambda}^A{}_B$, and γ_{AB}.

In Section 4 we calculate the Moffat–Ricci curvature scalar for $\tilde{W}^A{}_B$,

$$R(\tilde{W}) = \gamma^{AB}(R^C{}_{ABC}(\tilde{W}) + \tfrac{1}{2}R^C{}_{CAB}(\tilde{W})) \tag{10.8}$$

where $R^A{}_{BCD}(\tilde{W})$ is the tensor of curvature for the connection $\tilde{W}^A{}_B$, and we get

$$R(\tilde{W}) = \tilde{\tilde{R}}(\tilde{\tilde{W}}) + \frac{8\pi G_N}{c^4}\left\{\frac{1}{8\pi}\,[2(g^{[\mu\nu]}F_{\mu\nu})^2 - H^{\mu\alpha}F_{\mu\alpha}]\right\} \tag{10.9}$$

In (10.9) we come back to a normal system of physical units and in place of $\lambda=2$, we put $\lambda=2l_{\rm pl}/\alpha_{\rm em}$. Here $\bar{R}(\bar{W})$ is the Moffat–Ricci curvature scalar for the connection $\bar{W}^{\alpha}{}_{\beta}$,[34,63] and

$$\begin{aligned}\bar{\bar{R}} &= \mathbf{g}^{\alpha\beta}(\bar{\bar{R}}^{\gamma}{}_{\alpha\beta\gamma}(\bar{\bar{W}})+\tfrac{1}{2}\bar{\bar{R}}^{\gamma}{}_{\gamma\alpha\beta}(\bar{\bar{W}})) \\ &= \bar{\bar{R}}(\bar{\bar{\Gamma}})+\tfrac{2}{3}\mathbf{g}^{[\alpha\beta]}\bar{\bar{W}}_{[\alpha\beta]}\end{aligned} \tag{10.10}$$

where $\bar{\bar{R}}^{\alpha}{}_{\beta\gamma\delta}(\bar{\bar{W}})$ is the tensor of curvature for the connection $\bar{\bar{W}}^{\alpha}{}_{\beta}$,

$$\bar{\bar{W}}_{[\alpha\beta]}=\tfrac{1}{2}(\bar{\bar{W}}_{\alpha,\beta}-\bar{\bar{W}}_{\beta,\alpha})$$

and

$$\bar{\bar{R}}(\bar{\bar{\Gamma}})=\mathbf{g}_{\alpha\beta}(\bar{\bar{R}}^{\gamma}{}_{\alpha\beta\gamma}(\bar{\bar{\Gamma}})+\tfrac{1}{2}\bar{\bar{R}}^{\gamma}{}_{\gamma\alpha\beta}(\bar{\bar{\Gamma}}))$$

is the Moffat–Ricci curvature scalar for the connection $\tilde{\bar{\omega}}^{\alpha}{}_{\beta}$ [$\bar{\bar{R}}^{\alpha}{}_{\beta\gamma\delta}(\bar{\bar{\Gamma}})$ is the curvature tensor for the connection $\tilde{\bar{\omega}}^{\alpha}{}_{\beta}$].

Let us introduce material sources: a tensor of energy-momentum $\underset{\sim}{T}^{\mu\nu}$, a fermion current $\underset{\sim}{S}^{\mu}$, an electric current $\underset{\sim}{j}^{\alpha}$, and a phenomenological Lagrangian of material sources:

$$L_m=-\frac{8\pi G_N}{c^4}\mathbf{g}^{\mu\nu}\underset{\sim}{T}_{\mu\nu}+\frac{8\pi a^2}{3}\bar{\bar{W}}_{\mu}\underset{\sim}{S}^{\mu}+\frac{4\pi}{c}\underset{\sim}{j}^{\mu}A_{\mu} \tag{10.11}$$

$$\begin{aligned}\underset{\sim}{T}_{\mu\nu} &= -\frac{c^4}{8\pi G_N}\frac{\delta L_m}{\delta\mathbf{g}^{\mu\nu}} \\ \underset{\sim}{S}^{\mu} &= \frac{3}{8\pi a^2}\frac{\delta L_m}{\delta\bar{\bar{W}}_{\mu}} \\ \underset{\sim}{j}^{\mu} &= \frac{c}{4\pi}\frac{\delta L_m}{\delta A_{\mu}}\end{aligned} \tag{10.12}$$

and A_μ is the four-potential of the electromagnetic field. We assume that L_m is gauge invariant. This means that $\underset{\sim}{j}^{\mu}$ is conserved,

$$\partial_\mu\underset{\sim}{j}^{\mu}=0 \tag{10.13}$$

Let us recall some properties of the fermion current introduced by Moffat. Fermion current (fermion charge) plays the role of the second gravitational charge in NGT. This quantity has an influence on the geometry of space-time and it is conserved. In the nonsymmetric theory of gravitation, the fermion current is introduced in a phenomenological way:

$$\underset{\sim}{S}^{\mu}=\sum_i f_i^2\rho_i\underset{\sim}{u}^{\mu}$$

where f_i is the coupling constant of the ith fermion (with dimension of length) and ρ_i the density of the ith fermion.

We have

$$F = l^2 = \int \underset{\sim}{S}^4 \, d^3x = \sum_i f_i^2 N_i$$

and

$$\frac{dF}{dt} = \frac{dl^2}{dt} = 0 \qquad (F \text{ is the fermion number})$$

where $N_i = \int \rho_i u^4 \sqrt{-\mathrm{g}}\, d^3x$ denotes the number of the ith fermion.

In (10.11) we introduce a phenomenological Lagrangian with a term $(8\pi a^2/3)\bar{\bar{W}}_\mu \underset{\sim}{S}^\mu$, where a^2 is a universal coupling constant for a fermion. This constant is equal to one of the f_i^2 or a combination of them such that

$$\frac{8\pi}{3} \bar{\bar{W}}_\mu \underset{\sim}{S}^\mu = \frac{8\pi}{3} \bar{\bar{W}}_\mu \sum f_i^2 \rho_i \underset{\sim}{u}^\mu$$

In the case of only one kind of fermion we have $a^2 = f^2$. Thus, we can write this term as

$$\frac{8\pi}{3} \bar{\bar{W}}_\mu \underset{\sim}{S}^\mu = \frac{8\pi}{3} f^2 \rho \underset{\sim}{u}^\mu \bar{\bar{W}}_\mu$$

Let us define the Palatini variational principle on the manifold $\underline{P}$ for the density $[\sqrt{\gamma}\, R(\bar{\bar{W}}) + L_m]$

$$\delta \int_V [\sqrt{\gamma}\, R(\bar{\bar{W}}) + L_m]\, d^5x = 0, \qquad V \subset \underline{P} \tag{10.14}$$

where $\gamma = \det(\gamma_{AB}) = -\det(\mathrm{g}_{\alpha\beta}) = -\mathrm{g}$. We vary with respect to the independent quantities $\mathrm{g}_{\alpha\beta}$, $\bar{\bar{W}}^\alpha{}_{\beta\gamma}$, and A_μ. After simple calculations we get

$$\bar{\bar{R}}_{\alpha\beta}(\bar{\bar{W}}) - \tfrac{1}{2}\mathrm{g}_{\alpha\beta}\bar{\bar{R}}(\bar{\bar{W}}) = \frac{8\pi G}{c^4}(T_{\alpha\beta} + \overset{\text{em}}{T}_{\alpha\beta}) \tag{10.15}$$

$$\mathrm{g}^{[\mu\nu]}{}_{,\nu} = 4\pi a^2 \underset{\sim}{S}^\mu \tag{10.16}$$

$$\mathrm{g}_{\mu\nu,\sigma} - \mathrm{g}_{\rho\nu}\bar{\bar{\Lambda}}^\rho{}_{\mu\sigma} - \mathrm{g}_{\mu\rho}\bar{\bar{\Lambda}}^\rho{}_{\sigma\nu} = 0 \tag{10.17}$$

$$\partial_\mu \underset{\sim}{H}^{\alpha\mu} = \frac{4\pi}{c}\left[\underset{\sim}{j}^\alpha + 4a^2 c\, \underset{\sim}{S}^\alpha (\mathrm{g}^{[\mu\nu]} F_{\mu\nu}) + \frac{c}{2\pi} \underset{\sim}{\mathrm{g}}^{[\alpha\beta]}\, \partial_\beta(\mathrm{g}^{[\mu\nu]} F_{\mu\nu})\right] \tag{10.18}$$

where $\tilde{\tilde{\Lambda}}^{\rho}{}_{\mu\sigma}$ is the connection from Moffat's theory of gravitation[34] and

$$\tilde{\tilde{\Lambda}}^{\rho}{}_{\mu\sigma}=\tilde{\tilde{\Gamma}}^{\rho}{}_{\mu\sigma}+D^{\rho}{}_{\mu\sigma}(S) \qquad (10.19)$$

where

$$g_{\rho\nu}D^{\rho}{}_{\mu\sigma}+g_{\mu\rho}D^{\rho}{}_{\sigma\nu}=\frac{4\pi a^2}{3}S^{\rho}(g_{\mu\sigma}g_{\rho\nu}-g_{\mu\rho}g_{\sigma\nu}+g_{\mu\nu}g_{[\sigma\rho]}) \qquad (10.20)$$

Equations (10.15) and (10.16) are equations for the gravitational field in the presence of material and electromagnetic sources.

$T^{\text{em}}_{\alpha\beta}$ is the energy-momentum tensor for the electromagnetic field. Equation (10.17) is a compatability condition for the metric on space-time and it is usually satisfied in Moffat's theory of gravitation[34] if the fermion current is not zero. Equation (10.18) plays the role of the second Maxwell equation. Now we have on the right-hand side of (10.18) a sum of three currents: $\underset{\sim}{j}^{\alpha}$, $(c/2\pi)\underset{\sim}{g}^{[\alpha\beta]}\partial_{\beta}(g^{[\mu\nu]}F_{\mu\nu})$, and $4\pi a^2 \underset{\sim}{S}^{\alpha}(g^{[\alpha\beta]}F_{\mu\nu})$. The first is the current of external sources, the second is that known from the nonsymmetric Kaluza–Klein theory (see Section 9), and the third is induced by the fermion current. The total electric current

$$\overset{\text{tot}}{\underset{\sim}{j}^{\alpha}}=\underset{\sim}{j}^{\alpha}+\frac{c}{2\pi}\underset{\sim}{g}^{[\alpha\beta]}\,\partial_{\beta}(g^{[\mu\nu]}F_{\mu\nu})+4\pi a^2\underset{\sim}{S}^{\alpha}(g^{[\mu\nu]}F_{\mu\nu}) \qquad (10.21)$$

is conserved,

$$\partial_{\alpha}\overset{\text{tot}}{\underset{\sim}{j}^{\alpha}}=0 \qquad (10.22)$$

Let us define the tensor of the electromagnetic polarization $M_{\alpha\beta}$,

$$H_{\alpha\beta}=F_{\alpha\beta}-\frac{4\pi}{c}M_{\alpha\beta} \qquad (10.23)$$

It is easy to see that

$$Q^{5}{}_{\alpha\beta}(\tilde{\Gamma})=Q^{5}{}_{\alpha\beta}(\tilde{\Lambda})=\frac{8\pi}{c}M_{\alpha\beta} \qquad (10.24)$$

where $Q^{5}{}_{\alpha\beta}(\tilde{\Gamma})$ is the tensor of torsion in the fifth dimension for the connection $\tilde{\omega}^{A}_{B}$ and $Q^{5}{}_{\alpha\beta}(\tilde{\Lambda})$ is the tensor of torsion in the fifth dimension for the connection $\tilde{\Lambda}^{A}{}_{B}$. For the connection $\tilde{\Lambda}^{A}{}_{B}$ we have the compatibility condition (10.2). Thus, we get a compatibility condition for $\tilde{\Lambda}^{A}{}_{B}$ and an interpretation of the electromagnetic polarization as the torsion in the fifth dimension for the connection $\tilde{\Lambda}^{A}{}_{B}$. If $\underset{\sim}{S}^{\alpha}=0$, we get $\tilde{\omega}^{A}{}_{B}=\tilde{\Lambda}^{A}{}_{B}$.

11. SPIN SOURCES

Let us introduce spin sources into the phenomenological Lagrangian (10.11). To do this, we define on E (as in Ref. 74) two connections $\bar{W}$ and $\bar{\Gamma}$:

$$\bar{W}^{\alpha}{}_{\beta\gamma} = \bar{\tilde{W}}^{\alpha}{}_{\beta\gamma} + \kappa^{\alpha}{}_{\beta\gamma} \tag{11.1}$$

$$\bar{\Gamma}^{\alpha}{}_{\beta\gamma} = \bar{\tilde{\Gamma}}^{\alpha}{}_{\beta\gamma} + \kappa^{\alpha}{}_{\beta\gamma} \tag{11.2}$$

where $\kappa^{\alpha}{}_{\beta\gamma}$ is a tensor field such as

$$\kappa^{\alpha}{}_{\beta\alpha} = 0, \qquad \kappa^{\alpha}{}_{\beta\gamma} = -\kappa^{\alpha}{}_{\gamma\beta} \tag{11.3}$$

$\kappa^{\alpha}{}_{\beta\gamma}$ plays the role of the generalized contortion tensor from Einstein–Cartan theory.

It is easy to see that

$$\bar{W}^{\lambda}{}_{\mu\nu} = \bar{\Gamma}^{\lambda}{}_{\mu\nu} - \tfrac{2}{3}\delta^{\lambda}_{\mu}\bar{W}_{\nu} \tag{11.4}$$

where

$$\bar{W}_{\nu} = \tfrac{1}{2}(\bar{W}^{\sigma}{}_{\nu\sigma} - \bar{W}^{\sigma}{}_{\sigma\nu}) = \tfrac{1}{2}(\bar{\tilde{W}}^{\sigma}{}_{\nu\sigma} - \bar{\tilde{W}}^{\sigma}{}_{\sigma\nu}) = \bar{\tilde{W}}_{\nu}$$

We have

$$\bar{Q}^{\lambda}{}_{\mu\lambda}(\bar{\Gamma}) = \bar{\tilde{Q}}^{\lambda}{}_{\mu\lambda}(\bar{\tilde{\Gamma}}) = 0 \tag{11.5}$$

where $\bar{Q}^{\lambda}{}_{\mu\nu}(\bar{\Gamma})$ is the tensor of torsion for the connection $\bar{\Gamma}^{\lambda}{}_{\mu\nu}$ and $\bar{\tilde{Q}}^{\lambda}{}_{\mu\nu}(\bar{\tilde{\Gamma}})$ is the tensor of torsion for the connection $\bar{\tilde{\Gamma}}^{\lambda}{}_{\mu\nu}$. (See Ref. 75 for more details.)

Let us define connections $W^{A}{}_{B}$ and $\omega^{A}{}_{B}$ on P such that

$$W^{A}{}_{B} = \omega^{A}{}_{B} - \tfrac{4}{9}\delta^{A}_{B}\bar{W} \tag{11.6}$$

$$\omega^{A}{}_{B} = \left(\begin{array}{c|c} \pi^{*}(\bar{\omega}^{\alpha}{}_{\beta}) + g^{\gamma\alpha}H_{\gamma\beta}\theta^{5} & H_{\beta\gamma}\theta^{\gamma} \\ \hline g^{\alpha\beta}(H_{\gamma\beta} + 2F_{\beta\gamma})\theta^{\gamma} & 0 \end{array}\right) \tag{11.7}$$

and

$$\bar{W}^{\alpha}{}_{\beta} = \bar{W}^{\alpha}{}_{\beta\gamma}\bar{\theta}^{\gamma}, \quad \bar{\omega}^{\alpha}{}_{\beta} = \bar{\Gamma}^{\alpha}{}_{\beta\gamma}\bar{\theta}^{\gamma}, \qquad \bar{W} = \bar{W}_{\mu}\bar{\theta}^{\mu} \tag{11.8}$$

We define also the third connection

$$\Lambda^{A}{}_{B} = \left(\begin{array}{c|c} \pi'(\Omega^{\alpha}{}_{\beta}) + g^{\gamma\alpha}H_{\gamma\beta}\theta^{5} & H_{\beta\gamma}\bar{\theta}^{\gamma} \\ \hline g^{\alpha\beta}(H_{\gamma\beta} + 2F_{\beta\gamma})\bar{\theta}^{\gamma} & 0 \end{array}\right) \tag{11.9}$$

where $\Omega^{\alpha}{}_{\beta} = \Omega^{\alpha}{}_{\beta\gamma}\bar{\theta}^{\gamma}$ is a connection on space-time E such that

$$g_{\mu\nu,\sigma} - g_{\rho\nu}\Omega^{\rho}{}_{\mu\sigma} - g_{\mu\rho}\Omega^{\rho}{}_{\sigma\nu} = 0 \tag{11.10}$$

It is easy to see that $\Lambda^{A}{}_{B}$ satisfies the compatibility condition (10.2). Using formulas from Sections 4–9 and Ref. 34, one easily finds the Moffat–Ricci curvature scalar for the connection $W^A{}_B$:

$$R(W)=\bar{R}(\bar{W})+\frac{8\pi G_N}{c^4}\left\{\frac{1}{8\pi}\left[2(\mathbf{g}^{[\mu\nu]}F_{\mu\nu})^2-H^{\mu\alpha}F_{\mu\alpha}\right]\right\}$$

$$=\tilde{\bar{R}}(\tilde{\bar{\Gamma}})+\mathbf{g}^{\mu\alpha}\kappa^{\beta+}{}_{\mu+\alpha-;\beta}-\mathbf{g}^{\mu\alpha}\kappa^{\beta}{}_{\gamma\alpha}\kappa^{\gamma}{}_{\mu\beta}$$

$$+\tfrac{2}{3}\mathbf{g}^{[\mu\alpha]}W_{[\mu,\alpha]}+\frac{8\pi G_N}{c^4}\left\{\frac{1}{8\pi}\left[2(\mathbf{g}^{[\mu\nu]}F_{\mu\nu})^2-H^{\mu\alpha}F_{\mu\alpha}\right]\right\} \qquad (11.11)$$

where $\bar{R}(\bar{W})$ is the Moffat–Ricci curvature scalar for the connection $\bar{W}^{\alpha}{}_{\beta\gamma}$, and $\tilde{\bar{R}}(\tilde{\bar{\Gamma}})$ is the Moffat–Ricci curvature scalar for the connection $\tilde{\bar{\Gamma}}^{\alpha}{}_{\beta\gamma}$. Let us define the Lagrangian for the material sources such that

$$L'_m=L_m+\frac{8\pi G_N}{c^4}\bar{W}^{\sigma}{}_{\mu\nu}\cdot \underset{\sim}{S}_{\sigma}{}^{\mu\nu} \qquad (11.12)$$

[see (10.11)], where we put in place of $\tilde{\bar{W}}_\mu$ a vector $\bar{W}_\mu$ which is really equal to $\bar{W}_\mu$. We have (10.12) for L'_m and

$$\underset{\sim}{S}_{\sigma}{}^{\mu\nu}=\frac{8\pi G_N}{c^4}\frac{\delta L'_m}{\delta\bar{W}^{\sigma}{}_{\mu\nu}},\qquad \underset{\sim}{S}_{\sigma}{}^{\mu\nu}=-\underset{\sim}{S}_{\sigma}{}^{\nu\mu} \qquad (11.13)$$

for $\underset{\sim}{S}^{\mu}$ we have

$$\underset{\sim}{S}^{\mu}=\frac{8\pi G_N}{c^4}\frac{\delta L'_m}{\delta\tilde{\bar{W}}_{\mu}} \qquad (11.14)$$

Let us define the Palatini variational principle on the manifold $\underline{P}$:

$$\delta\int_V\left[L'_m+\sqrt{\gamma}\,R(W)\right]d^5x=0,\qquad V\subset P \qquad (11.15)$$

We vary with respect to the independent quantities $\mathbf{g}_{\mu\nu}$, $\bar{W}^{\lambda}{}_{\mu\nu}$, and A_μ. After some calculations we get

$$\bar{R}_{\mu\nu}(\bar{W})-\tfrac{1}{2}\mathbf{g}_{\mu\nu}\bar{R}(\bar{W})=\frac{8\pi G_N}{c^4}\left(T_{\mu\nu}+\overset{\text{em}}{T}_{\mu\nu}\right) \qquad (11.16)$$

$$\underset{\sim}{\mathbf{g}}^{[\mu\nu]}{}_{,\nu}=4\pi\left(a^2\underset{\sim}{S}^{\mu}-\frac{2G}{c^3}\underset{\sim}{S}_{\nu}{}^{\mu\nu}\right)=4\pi a^2\underset{\sim}{K}^{\mu} \qquad (11.17)$$

$$\mathbf{g}_{\mu\nu,\sigma}-\mathbf{g}_{\rho\nu}\tilde{\bar{\Lambda}}^{\rho}{}_{\nu\delta}-\mathbf{g}_{\mu\rho}\tilde{\bar{\Lambda}}^{\pi}{}_{\sigma\nu}$$

$$=\left(\mathbf{g}_{\rho\nu}\kappa^{\rho}{}_{\mu\sigma}+\mathbf{g}_{\mu\rho}\kappa^{\rho}{}_{\sigma\nu}+\frac{8\pi G_N}{c^3}\mathbf{g}_{\rho\nu}\mathbf{g}_{\mu\gamma}S_{\sigma}{}^{\rho\gamma}\right) \qquad (11.18)$$

$$\partial_\mu \underset{\sim}{H}^{\alpha\mu} = \frac{4\pi}{c}[\underset{\sim}{j}^{\alpha} + 4a^2 c \underset{\sim}{K}^{\alpha}(\underset{\sim}{\mathbf{g}}^{[\mu\nu]}F_{\mu\nu})] + \frac{c}{\pi}\mathbf{g}^{[\alpha\beta]}\partial_\beta(\underset{\sim}{\mathbf{g}}^{[\mu\nu]}F_{\mu\nu}) \quad (11.19)$$

To be in line with the usual interpretation of the Moffat compatibility condition, we suppose that

$$\mathbf{g}_{\mu\nu,\sigma} - \mathbf{g}_{\rho\nu}\tilde{\bar{\Lambda}}^{\rho}{}_{\mu\sigma} - \mathbf{g}_{\mu\rho}\tilde{\bar{\Lambda}}^{\rho}{}_{\sigma\mu} = 0 \quad (11.20)$$

and we get

$$\mathbf{g}_{\rho\nu}\kappa^{\rho}_{\mu\sigma} + \mathbf{g}_{\mu\rho}\kappa^{\rho}{}_{\sigma\nu} = -\left(\frac{8\pi G_N}{c^3}\right)\mathbf{g}_{\rho\nu}\mathbf{g}_{\mu\gamma}S_{\sigma}{}^{\rho\gamma} \quad (11.21)$$

i.e., a generalization of the Cartan equation from the Einstein–Cartan–Moffat theory.[74]

Equations (11.17) and (11.19) differ from the analogous equations from Section 10 [(10.16), (10.17)]. The tensorial density $\underset{\sim}{S}_{\sigma}{}^{\mu\nu}$ is a spin density and for a microscopic spin density (of a Dirac field or Rarita–Schwinger field) we have

$$\underset{\sim}{S}_{\nu}{}^{\mu\nu} = 0 \quad (11.22)$$

In the case of Mathesson spin (hydrodynamic macroscopic spin) one easily checks the same. We have $\underset{\sim}{S}_{\sigma}{}^{\mu\nu} = u_\sigma \underset{\sim}{S}^{\mu\nu}$, $u_\nu \underset{\sim}{S}^{\mu\nu} = 0$, $\underset{\sim}{S}^{\mu\nu} = -\underset{\sim}{S}^{\nu\mu}$, u_σ is the four-velocity of the fluid, and $\mathbf{S}^{\mu\nu}$ is a spin density tensor in the rest frame. Thus, we get

$$\underset{\sim}{\mathbf{g}}^{[\mu\nu]}{}_{,\nu} = 4\pi a^2 \underset{\sim}{S}^{\mu} \quad (11.23)$$

$$\partial_\mu \underset{\sim}{H}^{\alpha\mu} = \frac{4\pi}{c}\overset{\text{tot}}{\underset{\sim}{j}}{}^{\alpha} \quad (11.24)$$

where $\underset{\sim}{j}^{\text{tot},\alpha}$ is defined by (10.21).

Using equations (1.10) and (1.13) from Ref. 75, one transforms (11.16) into

$$\tilde{\bar{R}}_{\mu\alpha}(\tilde{\bar{W}}) - \tfrac{1}{2}\mathbf{g}_{\mu\alpha}\tilde{\bar{R}}(\tilde{\bar{W}}) = \frac{8\pi G_N}{c^4}(\overset{\text{em}}{T}_{\mu\alpha} + \overset{\text{eff}}{T}_{\mu\alpha}) \quad (11.25)$$

where

$$\overset{\text{eff}}{T}_{\mu\alpha} = T_{\mu\alpha} - \frac{c^4}{8\pi G_N}[\kappa^{\beta+}{}_{\alpha-\mu+;\beta} - \kappa^{\beta}{}_{\mu\delta}\kappa^{\delta}{}_{\alpha\beta} - \tfrac{1}{2}\mathbf{g}_{\mu\alpha}\mathbf{g}^{\gamma\nu}(\kappa^{\beta+}{}_{\nu-\gamma+;\beta} - \kappa^{\beta}{}_{\nu\delta}\kappa^{\delta}{}_{\gamma\beta})] \quad (11.26)$$

and $\kappa^{\alpha}{}_{\beta\gamma}$ is defined by (11.21). Thus, we get spin–spin interaction corrections from the Einstein–Cartan–Moffat theory.

Now it is easy to see that

$$\Omega^{\lambda}{}_{\mu\nu} = \tilde{\tilde{\Lambda}}^{\lambda}{}_{\mu\nu} \tag{11.27}$$

and

$$\Lambda^{A}{}_{B} = \tilde{\Lambda}^{A}{}_{B} \tag{11.28}$$

and the connection $\tilde{\Lambda}^{A}{}_{B}$ satisfies the compatibility condition for γ_{AB} on the manifold $\underset{\sim}{P}$. *For the polarization tensor* $M_{\alpha\beta}$ we have the same geometrical interpretation as before [see equation (10.24)].

12. GEODETIC EQUATIONS IN THE CASE OF NONZERO SOURCES

In the nonsymmetric Kaluza–Klein theory (Section 9) we have the following equation for geodesics:

$$\begin{gathered}\frac{\tilde{\tilde{D}}u^{\alpha}}{d\tau} + 2u^5(\mathrm{g}^{\alpha\gamma}F_{\gamma\beta} + \mathrm{g}^{[\alpha\gamma]}H_{\gamma\beta})u^{\beta} = 0 \\ u^5 = \text{const} \qquad \left(2u^5 = \frac{q}{m_0}\right)\end{gathered} \tag{12.1}$$

where q is the charge and m_0 is the rest mass of a test particle. Here $\tilde{\tilde{D}}/d\tau$ means covariant derivative with respect to $\tilde{\tilde{\omega}}^{\alpha}{}_{\beta}$ along a curve to which $u^{\alpha}(\tau)$ is tangent.

The usual interpretation of the geodetic equation in the Kaluza–Klein theory is that equation (12.1), after taking a local section of the electromagnetic bundle, is an equation of motion for a test particle in the gravitational and electromagnetic fields. Moreover, $F_{\mu\nu}$ and $H_{\mu\nu}$ are well defined on E and the shape of equation (12.1) does not change.

If we have nonzero fermion current $\underset{\sim}{S}^{\alpha} \neq 0$, it is necessary to put in place of $\tilde{\tilde{\omega}}^{\alpha}{}_{\beta}$ the connection $\tilde{\tilde{\Lambda}}^{\alpha}{}_{\beta}$ and we get in the holonomic system of coordinates

$$m_0\left(\frac{d^2x^{\alpha}}{d\tau^2} + \tilde{\tilde{\Lambda}}^{\alpha}{}_{(\beta\gamma)}\frac{dx^{\beta}}{d\tau}\frac{dx^{\gamma}}{d\tau}\right) + q(\mathrm{g}^{\alpha\gamma}F_{\gamma\beta} + \mathrm{g}^{[\alpha\gamma]}H_{\gamma\beta})\frac{dx^{\beta}}{d\tau} = 0 \tag{12.2}$$

The connection $\tilde{\tilde{\Lambda}}^{A}{}_{B}$ is compatible with the metric γ_{AB}. In Moffat's theory this kind of geodesic is called a nonextremal geodesic. Moreover, in Moffat's theory particles move along different geodesics,[34] i.e.,

$$\frac{d^2x^{\alpha}}{d\tau^2} + \begin{Bmatrix}\alpha \\ \beta\gamma\end{Bmatrix}\frac{dx^{\beta}}{d\tau}\frac{dx^{\gamma}}{d\tau} = 0 \tag{12.3}$$

Thus, we should put in place of $\tilde{\tilde{\Lambda}}{}^{\alpha}_{(\beta\gamma)}$ the Christoffel symbol $\{{}^{\alpha}_{\beta\gamma}\}$ for $g_{(\alpha\beta)}$. Finally, we get

$$m_0\left(\frac{d^2x^\alpha}{d\tau^2}+\left\{\begin{matrix}\alpha\\ \beta\gamma\end{matrix}\right\}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}\right)+q(g^{\alpha\gamma}F_{\gamma\beta}+g^{[\alpha\gamma]}H_{\gamma\beta})\frac{dx^\beta}{d\tau}=0 \qquad (12.4)$$

We will consider equation (12.4) the equation of motion for a test particle in the nonsymmetric Kaluza-Klein theory. The connection

$$\hat{\omega}^A{}_B=\left[\begin{array}{c|c}\pi^*(\{{}^{\alpha}_{\beta\gamma}\}\theta^\gamma)+g^{\gamma\alpha}H_{\gamma\beta}\theta^5 & H_{\beta\gamma}\theta^\gamma\\ \hline g^{\alpha\beta}(H_{\gamma\beta}+2F_{\beta\gamma})\theta^\gamma & 0\end{array}\right] \qquad (12.5)$$

is not compatible with the metric γ_{AB} on $\underline{P}$, just as the connection $\tilde{\tilde{\omega}}{}^\alpha{}_\beta=\{{}^{\alpha}_{\beta\gamma}\}\theta^\gamma$ is not compatible with $g_{\alpha\beta}$ on E. In the theory with spin sources, particles without spin and fermion charge move along geodesics in $\tilde{\omega}^A{}_B$ (as supposed in Ref. 74). The problem of motion for spinning particles with fermion charge demands further investigation.

Let us consider the geodesic equations (12.2) and (12.4) (the equation of motion for the test particle) in more detail. In Section 9 we proved that equation (12.2) has the following first integral of motion:

$$g_{(\alpha\beta)}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}=\text{const} \qquad (12.6)$$

if $\tilde{\tilde{\Lambda}}{}^\alpha{}_{\beta\gamma}=\tilde{\tilde{\Gamma}}{}^\alpha{}_{\beta\gamma}$. In this way we are able to keep a normalization for the four-velocity during the motion, i.e.,

$$g_{(\alpha\beta)}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}=1 \qquad (12.7)$$

for

$$\tau\geq\tau_0 \qquad \text{if} \quad g_{(\alpha\beta)}u_0^\alpha u_0^\beta=1 \qquad (12.8)$$

where

$$u_0^\alpha=\left.\frac{dx^\alpha}{d\tau}\right|_{\tau=\tau_0}$$

i.e., we consider timelike trajectories of a test particle. For the null case we need a different condition, i.e., $g_{(\alpha\beta)}u_0^\alpha u_0^\beta=0$. In this case we put $m_0=q=0$. However, in general, $u^5\neq 0$ and this describes the coupling of a particle to the electromagnetic field.

In the case of nonzero fermion current we have equation (12.2). Using similar arguments as in Section 9, we can prove that for equation (12.2) we have the integral of motion

$$g_{(\alpha\beta)} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = \text{const} \tag{12.9}$$

and we are able to keep the normalization for the four-velocity. We suppose that const ≥ 0 because spacelike world-lines are unphysical. The additional term for the Lorentz force

$$q \cdot g^{[\alpha\gamma]} H_{\gamma\beta} \frac{dx^\beta}{d\tau} \tag{12.10}$$

plays the role of a reaction force for the nonholonomic constants

$$g_{(\alpha\beta)} u^\alpha u^\beta = 1 \tag{12.11}$$

Let us consider equation (12.4) and multiply both sides of (12.4) by $g_{(\alpha\rho)}\, dx^\rho/d\tau$. We get

$$\frac{m_0}{2} \frac{d}{d\tau}\left(g_{(\alpha\beta)} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \right) = 0, \qquad m_0 \neq 0 \tag{12.12}$$

Thus, equation (12.4) has the same first integral of motion as equation (12.2). This means that we are able to keep the normalization of the four-velocity during the motion, i.e., equation (11.10): we consider timelike world-lines of a test particle ($m_0 \neq 0$). In the case of a null-line we have $m_0 = q = 0$ and $g_{\alpha\beta} u^\alpha u^\beta = 0$. Moreover, u^5 can be nonzero. In this way the term (12.10) plays the role of a reaction force for nonholonomic constraints (12.11) in equation (12.2) as in equation (12.4) (for $m_0 \neq 0$ and $q \neq 0$).

Let us pass to equation (10.20) in the weak-field approximation. We have

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{12.13}$$

where

$$|h_{\mu\nu}| < \alpha \ll 1 \tag{12.14}$$

For the inverse tensor we have

$$g^{\mu\nu} = \eta^{\mu\nu} + \tilde{h}^{\mu\nu} \cong \eta^{\mu\nu} - \eta^{\alpha\mu}\eta^{\gamma\nu} h_{\gamma\alpha} \tag{12.14a}$$

Let us rewrite equation (10.20) in a more convenient form,

$$\begin{aligned} & g_{\rho\nu}D^{\rho}{}_{\mu\sigma}+g_{\rho\mu}D^{\rho}{}_{\sigma\nu} \\ & \quad =-\frac{4\pi a^2}{3}S^{\rho}(g_{\mu\sigma}g_{\rho\nu}-g_{\mu\rho}g_{\sigma\nu}+g_{\mu\nu}g_{[\sigma\rho]})+g_{[\mu\rho]}D^{\rho}{}_{\sigma\nu} \end{aligned} \tag{12.15}$$

Let us consider the equation

$$g_{\rho\nu}\overset{(0)}{D}{}^{\rho}{}_{\mu\sigma}+g_{\rho\mu}\overset{(0)}{D}{}^{\rho}{}_{\sigma\nu}=-\frac{4\pi a^2}{3}S^{\rho}(g_{\mu\sigma}g_{\rho\nu}-g_{\mu\rho}g_{\sigma\nu}+g_{\mu\nu}g_{[\sigma\rho]}) \tag{12.16}$$

The solution of equation (12.16) is

$$\begin{aligned} \overset{(0)}{D}{}^{\alpha}{}_{\mu\sigma}=-\frac{2\pi a^2}{3}g^{\alpha\nu}S^{\rho}(&2g_{\mu\sigma}g_{[\rho\nu]}-2g_{\sigma\nu}g_{(\mu\rho)}+2g_{\nu\mu}g_{(\rho\sigma)} \\ &-g_{\mu\nu}g_{[\sigma\rho]}+g_{\nu\sigma}g_{[\mu\rho]}-g_{\sigma\mu}g_{[\nu\rho]}) \end{aligned} \tag{12.17}$$

Let us consider the following transformation:

$$\begin{aligned} & g_{\rho\nu}\overset{(n+1)}{D}{}^{\rho}{}_{\mu\sigma}+g_{\rho\mu}\overset{(n+1)}{D}{}^{\rho}{}_{\sigma\nu} \\ & \quad =-\frac{4\pi a^2}{3}S^{\rho}(g_{\mu\sigma}g_{\rho\nu}-g_{\mu\rho}g_{\rho\nu}+g_{\mu\nu}g_{[\rho\sigma]})+g_{[\mu\rho]}\overset{(n)}{D}{}^{\rho}{}_{\sigma\nu} \end{aligned} \tag{12.18}$$

for $n\geq 0$, or, after solving with respect to $\overset{(n+1)}{D}{}^{\rho}{}_{\sigma\nu}$,

$$\begin{aligned} \overset{(n+1)}{D}{}^{\alpha}{}_{\mu\sigma}=-\frac{2\pi a^2}{3}g^{\alpha\nu}S^{\rho}(&2g_{\mu\sigma}g_{[\rho\nu]}-2g_{\sigma\nu}g_{(\mu\rho)}+2g_{\nu\mu}g_{(\rho\sigma)} \\ &-g_{\mu\nu}g_{[\sigma\rho]}+g_{\nu\sigma}g_{[\mu\rho]}-g_{\sigma\mu}g_{[\nu\rho]}) \\ &+\tfrac{1}{2}g^{\alpha\nu}(g_{[\mu\rho]}\overset{(n)}{D}{}^{\rho}{}_{\sigma\nu}-g_{[\sigma\rho]}\overset{(n)}{D}{}^{\rho}{}_{\nu\mu}+g_{[\nu\rho]}\overset{(n)}{D}{}^{\rho}{}_{\mu\sigma}) \end{aligned} \tag{12.19}$$

for $n\geq 0$. The solution of equation (10.20) is a fix point of (12.18) or (12.19). Let us try to solve this equation using the iterative method based on (12.19).

Let us define

$$\overset{(n+1)}{D}{}^{\alpha}{}_{\mu\sigma}=N^{\alpha}{}_{\beta}{}^{\kappa}{}_{\mu}{}^{\delta}{}_{\sigma}\overset{(n)}{D}{}^{\beta}{}_{\kappa\delta} \tag{12.20}$$

such that for $n \geq 0$ we have (12.19) and $\overset{(n)}{D}{}^{\alpha}{}_{\mu\sigma}$ is expressed by (12.17). We get

$$\overset{(n+1)}{D}{}^{\alpha}{}_{\mu\sigma} - \overset{(n)}{D}{}^{\alpha}{}_{\mu\sigma} = \tfrac{1}{2}\mathbf{g}^{\alpha\nu}[\mathbf{g}_{[\mu\rho]}(\overset{(n)}{D}{}^{\rho}{}_{\sigma\nu} - \overset{(n-1)}{D}{}^{\rho}{}_{\sigma\nu}) + \mathbf{g}_{[\nu\rho]}(\overset{(n)}{D}{}^{\rho}{}_{\mu\sigma} - \overset{(n-1)}{D}{}^{\rho}{}_{\mu\sigma})$$
$$- \mathbf{g}_{[\sigma\rho]}(\overset{(n)}{D}{}^{\rho}{}_{\nu\mu} - \overset{(n-1)}{D}{}^{\rho}{}_{\nu\mu})] \tag{12.21}$$

for $n \geq 1$.

The tensors $D^{\rho}{}_{\mu\nu}$ form in a natural way the 64-dimensional linear (vector) space. Let us define a norm in this space:

$$\|D\| = \max_{\rho,\mu,\nu=1,2,3,4} |D^{\rho}{}_{\mu\nu}| \tag{12.22}$$

Now this space is a Banach space. Using (12.14) and (12.15), one gets

$$\|\overset{(n+1)}{D} - \overset{(n)}{D}\| < 24\alpha\left(1 + \frac{1}{1-4\alpha}\right)\|\overset{(n)}{D} - \overset{(n-1)}{D}\|$$
$$< 72\alpha\|\overset{(n)}{D} - \overset{(n-1)}{D}\| \tag{12.23}$$

If $\alpha < 1/72$, the transformation (12.20) is a contraction and according to the Banach theorem it has one and only one fix point such that

$$D^{\alpha}{}_{\mu\sigma} = \lim_{n\to\infty} (N^{n})^{\alpha}{}_{\beta}{}^{\rho}{}_{\mu}{}^{\delta}{}_{\sigma}\overset{(0)}{D}{}^{\beta}{}_{\rho\delta} \tag{12.24}$$

where N^{n} is nth interaction of the transformation (12.20). The limit is understood to be with respect to the norm (12.22).

Now we can write (12.24) in the form

$$D^{\alpha}{}_{\mu\sigma} = \overset{(\infty)}{N}{}^{\alpha}{}_{\beta}{}^{\kappa}{}_{\mu}{}^{\gamma}{}_{\sigma}\overset{(0)}{D}{}^{\beta}{}_{\kappa\delta} \tag{12.25}$$

where

$$\overset{(\infty)}{N}{}^{\alpha}{}_{\beta}{}^{\kappa}{}_{\mu}{}^{\delta}{}_{\sigma} = \lim_{n\to\infty} (N^{n})^{\alpha}{}_{\beta}{}^{\kappa}{}_{\mu}{}^{\delta}{}_{\sigma} \tag{12.26}$$

The last limit is understood in the sense of the usual linear operator topology generated by the topology of the Banach space. In this way we get, in the weak-field approximation,

$$\overset{\cong}{\Lambda}{}^{\alpha}{}_{\beta\gamma} = \overset{\cong}{\Gamma}{}^{\alpha}{}_{\beta\gamma} + D^{\alpha}{}_{\beta\gamma}$$
$$= \overset{\cong}{\Gamma}{}^{\alpha}{}_{\beta\gamma} - \frac{2\pi a^{2}}{3}\overset{(\infty)}{N}{}^{\alpha}{}_{\sigma}{}^{\kappa}{}_{\beta}{}^{\delta}{}_{\gamma}\mathbf{g}^{\sigma\nu}S^{\rho}(2\mathbf{g}_{\kappa\delta}\mathbf{g}_{[\rho\nu]}$$
$$- 2\mathbf{g}_{\delta\nu}\mathbf{g}_{(\rho\kappa)} + 2\mathbf{g}_{\nu\kappa}\mathbf{g}_{(\rho\sigma)} - \mathbf{g}_{\kappa\nu}\mathbf{g}_{[\delta\rho]}$$
$$+ \mathbf{g}_{\nu\delta}\mathbf{g}_{[\kappa\rho]} - \mathbf{g}_{\delta\kappa}\mathbf{g}_{[\nu\rho]}) \tag{12.27}$$

If we use (12.27) and (9.36), we get for (12.2)

$$m_0\left[\frac{d^2x^\alpha}{d\tau^2}+\overset{\approx}{\Gamma}{}^\alpha{}_{\beta\gamma}\left(\frac{dx^\alpha}{d\tau}\right)\left(\frac{dx^\gamma}{d\tau}\right)\right]-\frac{2\pi a^2 m_0}{3}\overset{(\infty)}{N^\alpha}{}_\sigma{}^\kappa{}_{(\beta}{}^\delta{}_{\gamma)}$$

$$\times\, \mathbf{g}^{\sigma\nu}S^\rho(2\mathbf{g}_{\kappa\delta}\mathbf{g}_{[\rho\nu]}-2\mathbf{g}_{\delta\nu}\mathbf{g}_{(\rho\kappa)}+2\mathbf{g}_{\nu\kappa}\mathbf{g}_{(\rho\delta)}$$

$$-\mathbf{g}_{\kappa\nu}\mathbf{g}_{[\delta\rho]}+\mathbf{g}_{\nu\delta}\mathbf{g}_{[\kappa\rho]}-\mathbf{g}_{\delta\kappa}\mathbf{g}_{[\nu\rho]})\left(\frac{dx^\beta}{d\tau}\right)\left(\frac{dx^\gamma}{d\tau}\right)$$

$$+q(\mathbf{g}^{\alpha\gamma}F_{\gamma\beta}-\mathbf{g}^{[\alpha\gamma]}\overset{(\infty)}{M^{\mu\nu}}{}_{\gamma\beta}F_{\mu\nu})\frac{dx^\beta}{d\tau}=0 \tag{12.28}$$

where $M^{(\infty)\mu\nu}{}_{\gamma\beta}$ is defined by equation (9.35). Because $M^{(\infty)}$ was considered as the limit in the weak-field approximation for $\alpha<1/96$, we must choose

$$\alpha<\min[1/72, 1/96]=1/96 \tag{12.29}$$

Let us pass to equation (12.4). In the weak-field approximation for $\alpha<1/96$ we get

$$m_0\left[\frac{d^2x^\alpha}{d\tau^2}+\left\{{\alpha \atop \beta\gamma}\right\}\left(\frac{dx^\beta}{d\tau}\right)\left(\frac{dx^\gamma}{d\tau}\right)\right]$$

$$+q\left(\mathbf{g}^{\alpha\gamma}F_{\gamma\beta}-\mathbf{g}^{[\alpha\gamma]}\overset{(\infty)}{M^{\mu\nu}}{}_{\gamma\beta}F_{\mu\nu}\cdot\frac{dx^\beta}{d\tau}\right)=0 \tag{12.30}$$

13. NUMERICAL PREDICTIONS OF THE THEORY

Let us pass to equation (8.4). We get here an additional term for the Lorentz force

$$\frac{q}{m_0}\mathbf{g}^{[\alpha\gamma]}H_{\gamma\beta}u^\beta \tag{13.1}$$

In the Moffat theory of gravitation there is an exact solution which is spherically symmetric and static (Schwarzschild-like solution). It has the following form[34]:

$$\mathbf{g}_{\mu\nu}=\begin{bmatrix}-(1-2m/r)^{-1} & 0 & 0 & l^2/r^2\\ 0 & -r^2 & 0 & 0\\ 0 & 0 & -r^2\sin^2\theta & 0\\ -l^2/r^2 & 0 & 0 & (1-2m/r)(1+l^4/r^4)\end{bmatrix} \tag{13.2}$$

For the inverse tensor $g^{\mu\nu}$ we have similarly

$$g^{\mu\nu}=\begin{bmatrix} -(1-2m/r)(1+l^4/r^4) & 0 & 0 & -l^2/r^2 \\ 0 & -1/r^2 & 0 & 0 \\ 0 & 0 & -1/(r^2\sin^2\theta) & 0 \\ l^2/r^2 & 0 & 0 & (1-2m/r)^{-1} \end{bmatrix} \tag{13.3}$$

where m is the mass and l^2 is the fermion charge. Let us estimate the contribution of (13.1) to the Lorentz force term on the surface of the sun, using (13.2) as a metric. In the Moffat theory[45]

$$l=l_\odot=(3.1-0.5)\times 10^3\ \mathrm{km} \tag{13.4}$$

and we have for the radius of the sun

$$R_\odot=0.7\times 10^6\ \mathrm{km} \tag{13.5}$$

Thus, on the surface of the sun we get

$$w_\odot=\frac{l_\odot^2}{R_\odot^2}\simeq 10^{-6} \tag{13.6}$$

If we consider equation (4.9) for (13.2), we get

$$H_{\beta\gamma}=F_{\beta\gamma} \tag{13.7}$$

We get

$$\frac{q}{m_0}g^{[\alpha\gamma]}H_{\gamma\beta}u^\beta=\frac{q}{m_0}g^{[\alpha\gamma]}F_{\gamma\beta}u^\beta \tag{13.8}$$

But the only nonvanishing component of $g^{[\alpha\gamma]}$ is

$$g^{[14]}=-w_\odot\simeq 10^{-6} \tag{13.9}$$

and the contribution (13.8) to the Lorentz force is 10^{-6} in comparison to the usual Lorentz force term. Thus, it is negligible in the solar system. However, for a neutron star we have[34]

$$l_N=7\ \mathrm{km},\qquad R_N=6\ \mathrm{km},\qquad w_N\approx 1 \tag{13.10}$$

and this new term should play a role. Unfortunately, only $g^{[14]}=w_N\neq 0$. Thus, we only have a new term for the electric part of the electromagnetic field. It is the same for the new term in the Lagrangian

$$2(g^{[\mu\nu]}F_{\mu\nu})^2=2w_N^2(F_{14})^2=2w_N^2E_N^2 \tag{13.11}$$

The electric field does not play any important role on the surface of neutron stars, in contrast to the magnetic field, and this does not contradict observational data.

It is interesting to ask if this statement will hold in the case of nonzero external sources. In order to do this, let us consider equation (10.18),

$$\partial_\mu \underset{\sim}{H}^{\mu\alpha} = \frac{4\pi}{c} \overset{\text{tot}}{\underset{\sim}{j}}{}^{\alpha} \tag{13.12}$$

where

$$\overset{\text{tot}}{\underset{\sim}{j}}{}^{\alpha} = \underset{\sim}{j}^{\alpha} + 2c\underset{\sim}{S}^{\alpha}(\mathbf{g}^{[\mu\nu]}F_{\mu\nu}) + \frac{c}{2\pi}\mathbf{g}^{[\alpha\beta]}\partial_\beta(\mathbf{g}^{[\mu\nu]}F_{\mu\nu}) \tag{13.13}$$

In equation (13.13) we get an addition effect, i.e., a new term in the total current

$$2\pi\underset{\sim}{S}^{\alpha}(\mathbf{g}^{[\mu\nu]}F_{\mu\nu}) \tag{13.14}$$

Let us estimate the influence of this term in the solar system and on the surface of a neutron star. In order to do this, we consider a simple model of dust with convective electric and fermion currents,

$$j^{\alpha} = cq\rho u^{\alpha} \tag{13.15}$$

$$S^{\alpha} = f^4 \rho u^{\alpha} \tag{13.16}$$

where u^α is the four-velocity of the dust, ρ is the density of the dust, q is the electric charge of a dust particle, and f^2 is its fermion charge (notice that the constant a^2 does not appear here because we consider only one kind of fermion and $a^2=f^2$). Let us consider an effective electric current on the surface of the sun,

$$\overset{\text{eff}}{j}{}^{\alpha} = j^{\alpha} + 2cS^{\alpha}(\mathbf{g}^{[\mu\nu]}F_{\mu\nu}) = u^{\alpha}c\rho\left(q - \frac{4f^2 l_\odot^2}{R_\odot^2}E_\odot\right) = cq_{\text{eff}}\rho u^{\alpha} \tag{13.17}$$

where

$$q_{\text{eff}} = q - \frac{4f^2 l_\odot^2}{R_\odot^2}E_\odot \tag{13.18}$$

In equation (13.17) we use that[34,65]

$$\mathbf{g}^{[14]} = -w_\odot = -\left(\frac{l_\odot}{R_\odot}\right)^2 \tag{13.19}$$

and

$$F_{14} = E_{\odot} \tag{13.20}$$

is the electric field on the surface of the sun, $l_{\odot}$ is the fermion number parameter for the sun from the nonsymmetric theory of gravitation, and $R_{\odot}$ is the radius of the sun. We also have[63]

$$l_{\odot}^2 = f^2 N_{\odot} \tag{13.21}$$

where $N_{\odot}$ is the number of protons for the sun. f^2 is here a universal constant (fermion charge for a nucleon). Using equation (13.21), one easily gets

$$q_{\text{eff}} = q - \frac{4 l_{\odot}^4 m_p}{R_{\odot}^2 M_{\odot}} E_{\odot} = q - \Delta q \tag{13.22}$$

where

$$N_{\odot} \simeq \frac{M_{\odot}}{m_p} \tag{13.23}$$

$M_{\odot}$ denotes the mass of the sun and m_p the mass of the proton. Let us estimate the contribution of Δq to q_{eff}:

$$\left|\frac{\Delta q}{q}\right|_{\odot} = \frac{4 l_{\odot}^2}{q} \frac{l_{\odot}^2}{R_{\odot}^2} \frac{m_p}{M_{\odot}} E_{\odot} \tag{13.24}$$

One easily gets

$$\left|\frac{\Delta q}{q}\right|_{\odot} \cong 6 \times 10^{-36} \frac{E_{\odot}}{[\text{esu/cm}^2]} \tag{13.25}$$

where

$$q \simeq 4.8 \times 10^{-36} \text{ esu (elementary charge)} \tag{13.26}$$

$$\frac{M_{\odot}}{m_p} \simeq 1.2 \times 10^{57} \tag{13.27}$$

If we put[75]

$$E_{\odot} \simeq 8 \times 10^6 [\text{esu/cm}^2] \tag{13.28}$$

we get

$$\left|\frac{\Delta q}{q}\right|_{\odot} \simeq 10^{-29} \tag{13.29}$$

Thus, it is completely negligible on the surface on the sun.

Let us come back to the formula (13.13) and estimate the value of an electric field for which we have screening of the electric charge. This means that

$$q_{\text{eff}} \cong 0 \tag{13.30}$$

and

$$E = E_{\text{scr}} = \frac{1}{4}\frac{M_\odot}{m_p}\left(\frac{R_\odot}{l_\odot}\right)\frac{q}{l_\odot} \cong 16 \times 10^{35}\left[\frac{\text{esu}}{\text{cm}^2}\right] \tag{13.31}$$

In this way

$$\overset{\text{eff}}{j^\alpha} \cong 0 \tag{13.32}$$

for $E = E_{\text{scr}}$.

Let us perform similar calculations for the surface of a neutron star. We get

$$q_{\text{eff}} = q - \frac{4 l_N^2 m_p}{R_N^2 M_\odot} E_N \tag{13.33}$$

We have for a neutron star[34,65]

$$w_N = \frac{l_N^2}{R_N^2} \simeq 1 \tag{13.34}$$

Thus,

$$\left|\frac{\Delta q}{q}\right|_N = \frac{4}{q}\frac{l_\odot^2 l_N^2}{R_N^2}\frac{m_p}{M_\odot} E_N \tag{13.35}$$

and we get

$$\left|\frac{\Delta q}{q}\right|_N \simeq 6 \times 10^{-30}\frac{E_N}{[\text{esu/cm}^2]} \tag{13.36}$$

If we put[76]

$$E_N \cong 0.33 \times 10^{10}\left[\frac{\text{esu}}{\text{cm}^2}\right] \tag{13.37}$$

we get

$$\left|\frac{\Delta q}{q}\right|_N \simeq 0.2 \times 10^{-19} \tag{13.38}$$

Thus, Δq is completely negligible on the surface of a neutron star. Let us estimate the value of an electric field for which we have screening of the electric charge, i.e.,

$$q_{\text{eff}} \cong 0 \tag{13.39}$$

We get

$$E_{\text{scr}} = \frac{q}{8} \frac{R_N^2}{l_\odot^2 l_N^2} \frac{M_\odot}{m_p} \cong 1.6 \times 10^{30} \left[\frac{\text{esu}}{\text{cm}^2}\right] \tag{13.40}$$

Similar to the case for the sun, equation (13.40) indicates

$$\overset{\text{tot}}{j^\alpha} \cong 0 \tag{13.41}$$

for $E = E_{\text{scr}}$.

Thus, the additional term for the current does not contradict any observational or experimental data for the solar system or for the surface of a neutron star.

Moreover, it is possible to predict significant effects by finding exact solutions of the full field equations. This is possible using the more general metric

$$g_{\mu\nu} = \begin{bmatrix} -\alpha & 0 & 0 & w \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin^2\theta & 0 \\ -w & 0 & 0 & \gamma \end{bmatrix} \tag{13.42}$$

where

$$\begin{aligned} \alpha &= \left[1 - \frac{2m}{r} + \beta(r)\right]^{-1} \\ w &= \frac{l^2}{r^2} \\ \gamma &= \left[1 - \frac{2m}{r} + \beta(r)\right]^{-1} \left(1 + \frac{l^4}{r^4}\right) \end{aligned} \tag{13.43}$$

Such solutions have been found and we discuss them in Sections 18–22.

14. SPIN SOURCES. WEAK-FIELD APPROXIMATION OF THE GENERALIZED CARTAN EQUATION

In Ref. 74 we deal with spin sources in NGT (nonsymmetric gravitation theory). In this way we construct an Einstein–Cartan–Moffat theory of gravitation.[74] We also consider spin sources in the nonsymmetric Kaluza–Klein theory in Section 11. We derive the generalized Cartan equation which connects the spin density tensor and the generalized contortion tensor. This equation is algebraic (generalized contortion does not propagate). We have

$$\begin{aligned} &\mathbf{g}_{\rho\nu}K^{\rho}{}_{\sigma\sigma}+\mathbf{g}_{\mu\rho}K^{\rho}{}_{\sigma\nu}=-\frac{8\pi G_N}{c^3}\mathbf{g}_{\rho\nu}\mathbf{g}_{\mu\gamma}S_{\sigma}{}^{\rho\gamma} \\ &K^{\rho}{}_{\mu\sigma}=-K^{\rho}{}_{\sigma\mu}, \qquad K^{\rho}{}_{\mu\rho}=0 \end{aligned} \tag{14.1}$$

Let us solve this equation in the weak-field approximation, i.e., let us suppose conditions (11.3). In order to do this, let us rewrite (14.1) in the following way:

$$\begin{aligned} \mathbf{g}_{\rho\nu}K^{\rho}{}_{\mu\sigma}+\mathbf{g}_{\rho\mu}K^{\rho}{}_{\sigma\nu}=&-\frac{8\pi G_N}{c^3}\mathbf{g}_{\rho\nu}\mathbf{g}_{\gamma\mu}S_{\sigma}{}^{\rho\gamma} \\ &-\left(\frac{8\pi G_N}{c^3}\mathbf{g}_{\rho\nu}\mathbf{g}_{[\mu\gamma]}S_{\sigma}{}^{\rho\gamma}+\mathbf{g}_{[\mu\rho]}K^{\rho}{}_{\sigma\nu}\right) \end{aligned} \tag{14.2}$$

Let us consider the following equation:

$$\mathbf{g}_{\rho\nu}\overset{(0)}{K}{}^{\rho}{}_{\mu\sigma}+\mathbf{g}_{\rho\gamma}\overset{(0)}{K}{}^{\rho}{}_{\sigma\nu}=-\frac{8\pi G_N}{c^3}\mathbf{g}_{\rho\nu}\mathbf{g}_{\gamma\mu}S_{\sigma}{}^{\rho\gamma} \tag{14.3}$$

where

$$\overset{(0)}{K}{}^{\rho}{}_{\mu\sigma}+\overset{(0)}{K}{}^{\rho}{}_{\sigma\mu}=0$$

The solution of equation (13.3) is as follows:

$$\overset{(0)}{K}{}^{\rho}{}_{\mu\sigma}=-\frac{1}{4}\frac{8\pi G_N}{c^3}(\mathbf{g}_{\gamma\mu}S_{\sigma}{}^{\rho\gamma}-\mathbf{g}_{\gamma\sigma}S_{\mu}{}^{\rho\gamma}) \tag{14.4}$$

Let us consider the following transformation (for $n\geq 0$):

$$\begin{aligned} \mathbf{g}_{\rho\nu}\overset{(n+1)}{K}{}^{\rho}{}_{\mu\sigma}+\mathbf{g}_{\rho\mu}\overset{(n+1)}{K}{}^{\rho}{}_{\sigma\nu}=&-\frac{8\pi G_N}{c^3}\mathbf{g}_{\rho\nu}\mathbf{g}_{\gamma\mu}S_{\sigma}{}^{\rho\gamma} \\ &-\left(\frac{8\pi G_N}{c^3}\mathbf{g}_{\rho\nu}\mathbf{g}_{[\mu\gamma]}S_{\sigma}{}^{\rho\gamma}+\mathbf{g}_{[\mu\rho]}\overset{(n)}{K}{}^{\rho}{}_{\sigma\nu}\right) \end{aligned} \tag{14.5}$$

or equivalently

$$\overset{(n+1)}{\kappa^{\rho}}{}_{\mu\sigma} = -\frac{1}{2}\frac{8\pi G_N}{c^3} g_{\rho\nu} g_{[\nu|\gamma|} S_{\sigma]}{}^{\rho\gamma} - \tfrac{1}{2}\left(g_{[\mu\rho]} \overset{(n)}{\kappa^{\rho}}{}_{\sigma\nu} - g_{[\sigma\rho]} \overset{(n)}{\kappa^{\rho}}{}_{\mu\nu}\right) \tag{14.6}$$

$$\overset{(n)}{\kappa^{\rho}}{}_{\sigma\nu} + \overset{(n)}{\kappa^{\rho}}{}_{\nu\sigma} = 0 \tag{14.6a}$$

for $n \geq 0$.

We get

$$\overset{(n+1)}{\kappa^{\rho}}{}_{\mu\sigma} - \overset{(n)}{\kappa^{\rho}}{}_{\mu\sigma} = -\tfrac{1}{2}\left[g_{[\mu\rho]}\left(\overset{(n)}{\kappa^{\rho}}{}_{\sigma\nu} - \overset{(n-1)}{\kappa^{\rho}}{}_{\sigma\nu}\right) - g_{[\sigma\rho]}\left(\overset{(n)}{\kappa^{\rho}}{}_{\mu\nu} - \overset{(n-1)}{\kappa^{\rho}}{}_{\mu\nu}\right)\right] \tag{14.7}$$

The skew-symmetric tensors $\kappa^{\alpha}{}_{\mu\nu}$ form a natural 24-dimensional linear (vector) space. Let us introduce the following norm in this space:

$$\|\kappa\| = \max_{\alpha,\mu,\nu=1,2,3,4} |\kappa^{\alpha}{}_{\mu\nu}| \tag{14.8}$$

Thus, this is a Banach space. Using (14.7), (14.8), and (11.3), one gets

$$\|\overset{(n+1)}{\kappa} - \overset{(n)}{\kappa}\| < 4\alpha \|\overset{(n)}{\kappa} - \overset{(n-1)}{\kappa}\| \tag{14.9}$$

Thus, the transformation (14.7) is a contraction if $\alpha < 1/4$ and according to the Banach theorem it has one and only one fix point such that

$$\kappa^{\rho}{}_{\mu\sigma} = \lim_{n\to\infty} (M^n)^{\alpha}{}_{\mu}{}^{\beta}{}_{\sigma}{}^{\rho}{}_{\gamma} \overset{(0)}{\kappa^{\gamma}}{}_{\alpha\beta} \tag{14.10}$$

where we have

$$\overset{(n+1)}{\kappa^{\rho}}{}_{\mu\sigma} = M^{\alpha}{}_{\nu}{}^{\beta}{}_{\sigma}{}^{\rho}{}_{\gamma} \overset{(n)}{\kappa^{\gamma}}{}_{\alpha\beta} = (M^{n+1})^{\alpha}{}_{\mu}{}^{\beta}{}_{\sigma}{}^{\rho}{}_{\gamma} \overset{(0)}{\kappa^{\gamma}}{}_{\alpha\beta} \tag{14.11}$$

for $n \geq 0$ defined by equation (14.7); $\kappa^{(0)\gamma}{}_{\alpha\beta}$ is expressed by equation (14.4). M^n is the nth iteration of the transformation (14.7). The limit (13.10) is understood to be with respect to the norm (14.8). We can write (14.10) in the form

$$\kappa^{\rho}{}_{\mu\sigma} = \overset{(\infty)}{M}{}^{\alpha}{}_{\mu}{}^{\beta}{}_{\sigma}{}^{\rho}{}_{\gamma} \overset{(0)}{\kappa^{\gamma}}{}_{\alpha\beta} \tag{14.12}$$

where

$$\overset{(\infty)}{M}{}^{\alpha}{}_{\mu}{}^{\beta}{}_{\sigma}{}^{\rho}{}_{\gamma} = \lim_{n\to\infty} (M^n)^{\alpha}{}_{\mu}{}^{\beta}{}_{\sigma}{}^{\rho}{}_{\gamma} \tag{14.13}$$

The last limit is understood in the sense of the usual linear operator topology generated by the topology of the Banach space. Thus, we get

$$\kappa^{\rho}{}_{\mu\sigma} = -\frac{8\pi G_N}{c^3} \overset{(\infty)}{M}{}^{\alpha}{}_{\mu}{}^{\beta}{}_{\sigma}{}^{\rho}{}_{\gamma} \mathbf{g}_{\gamma[\alpha} S_{\beta]}{}^{\gamma\delta} \tag{14.14}$$

In Section 11 we use the so-called effective energy-momentum tensor:

$$\overset{\text{eff}}{T}_{\mu\alpha} = T_{\mu\alpha} - \frac{c^4}{8\pi G_N} [\kappa^{\beta+}{}_{\alpha-\mu+;\beta} - \kappa^{\beta}{}_{\alpha\gamma}\kappa^{\gamma}{}_{\mu\beta} - \tfrac{1}{2}\mathbf{g}_{\mu\alpha}\mathbf{g}^{\gamma\nu}(\kappa^{\beta+}{}_{\nu-\gamma+;\beta} - \kappa^{\beta}{}_{\nu\delta}\kappa^{\delta}{}_{\gamma\beta})] \tag{14.15}$$

Using equation (14.14), one easily gets

$$\begin{aligned}\overset{\text{eff}}{T}_{\mu\alpha} = T_{\mu\alpha} + \frac{1}{2}\Bigg\{&(\overset{(\infty)}{M}{}^{\nu}{}_{\alpha-}{}^{\rho}{}_{\mu+}{}^{\beta+}{}_{\gamma}\mathbf{g}_{\delta[\nu}S_{\rho]}{}^{\gamma\delta})_{;\beta} \\ &+ \frac{4\pi G_N}{c^3}\overset{(\infty)}{M}{}^{\nu}{}_{\alpha}{}^{\rho}{}_{\gamma}{}^{\beta}{}_{\omega}\overset{(\infty)}{M}{}^{\psi}{}_{\mu}{}^{\phi}{}_{\beta}{}^{\gamma}{}_{\kappa}\mathbf{g}_{\delta[\nu}S_{\rho]}{}^{\omega\delta}\mathbf{g}_{\sigma[\psi}S_{\phi]}{}^{\kappa\sigma} \\ &= -\tfrac{1}{2}\mathbf{g}_{\mu\alpha}\mathbf{g}^{\gamma\nu}\Bigg[(\overset{(\infty)}{M}{}^{\psi}{}_{\gamma-}{}^{\phi}{}_{\nu+}{}^{\beta+}{}_{\sigma}\mathbf{g}_{\delta[\psi}S_{\phi]}{}^{\sigma\delta})_{;\beta} \\ &+ \frac{4\pi G_N}{c^3}\overset{(\infty)}{M}{}^{\psi}{}_{\nu}{}^{\phi}{}_{\kappa}{}^{\beta}{}_{\xi}\overset{(\infty)}{M}{}^{\omega}{}_{\gamma}{}^{\rho}{}_{\beta}{}^{\kappa}{}_{\rho}\mathbf{g}_{\delta[\psi}S_{\phi]}{}^{\xi\delta}\mathbf{g}_{\sigma[\omega}S_{\rho]}{}^{\rho\sigma}\Bigg]\Bigg\}\end{aligned} \tag{14.16}$$

Thus, we get a spin–spin contact interaction similar to that in Einstein–Cartan theory.[77] In the case of a weak field we can write in the place of the covariant derivative (semicolon) the partial derivative. We can omit terms which are quadratic with respect to the spin density tensor. In this case we can also use the zeroth-order approximation for $\kappa^{\rho}{}_{\mu\sigma}$, i.e.,

$$\kappa^{\rho}{}_{\mu\sigma} = \overset{(0)}{\kappa}{}^{\rho}{}_{\mu\sigma} = -\frac{2\pi G_N}{c^3}(\mathbf{g}_{\gamma\mu}S_{\sigma}{}^{\rho\gamma} - \mathbf{g}_{\gamma\sigma}S_{\mu}{}^{\rho\gamma}) \tag{14.17}$$

Thus, we can write

$$\begin{aligned}\overset{\text{eff}}{T}_{\mu\alpha} &= T_{\mu\alpha} + \frac{1}{4}\frac{\partial}{\partial X^{\beta}}(\mathbf{g}_{\gamma\alpha}S_{\mu}{}^{\beta\gamma} - \mathbf{g}_{\gamma\mu}S_{\alpha}{}^{\beta\gamma}) \\ &= T_{\mu\alpha} - \frac{1}{4}\frac{\partial}{\partial X^{\beta}}(S_{\alpha}{}^{\beta}{}_{\mu} - S_{\mu}{}^{\beta}{}_{\alpha}) + \frac{1}{4}\frac{\partial}{\partial X^{\beta}}(h_{(\gamma\alpha)}S_{\mu}^{\beta\gamma} - h_{(\gamma\mu)}S_{\alpha}^{\beta\gamma}) \\ &\quad + \frac{1}{4}\frac{\partial}{\partial X^{\beta}}(h_{[\gamma\alpha]}S_{\mu}{}^{\beta\gamma} - h_{[\gamma\mu]}S_{\alpha}{}^{\beta\gamma})\end{aligned} \tag{14.18}$$

where

$$S_{\alpha}{}^{\beta}{}_{\mu} = \eta_{\gamma\mu} S_{\alpha}{}^{\beta\gamma}$$

15. LINEARIZATION OF THE NONSYMMETRIC KALUZA–KLEIN (JORDAN–THIRY) THEORY IN THE ELECTROMAGNETIC CASE

Let us develop a linearization for the tensor $\mathbf{g}_{\mu\nu}$,

$$\mathbf{g}_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{15.1}$$

where $\eta_{\mu\nu}$ is a Minkowski tensor.[78,79] We have

$$\mathbf{g}^{\mu\alpha}\mathbf{g}_{\nu\alpha} = (\eta^{\mu\alpha} + \overset{1}{h}{}^{\mu\alpha} + \overset{2}{h}{}^{\mu\alpha} + \cdots)(\eta_{\nu\alpha} + h_{\nu\alpha}) = \delta^{\mu}_{\nu} \tag{15.2}$$

From (15.2) we get

$$\overset{1}{h}{}^{\mu\nu} = -\eta^{\mu\alpha}\eta^{\nu\beta}h_{\beta\alpha} \tag{15.3}$$

$$\overset{2}{h}{}^{\mu\nu} = -\eta^{\nu\beta}\overset{1}{h}{}^{\mu\alpha}h_{\beta\alpha} = \eta^{\nu\beta}\eta^{\mu\gamma}\eta^{\alpha\sigma}h_{\sigma\gamma}h_{\beta\alpha} \tag{15.4}$$

and

$$\mathbf{g}^{\mu\nu} = \eta^{\mu\nu} - \eta^{\mu\alpha}\eta^{\nu\beta}h_{\beta\alpha} + \eta^{\mu\gamma}\eta^{\nu\alpha}\eta^{\alpha\beta}h_{\beta\gamma}h_{\sigma\alpha} + \cdots \tag{15.5}$$

Let us write equation (4.9) in a more convenient form,

$$H_{\beta\alpha} - \mathbf{g}^{\gamma\delta}[\mathbf{g}_{[\beta\delta]}H_{\gamma\alpha} + \mathbf{g}_{[\gamma\alpha]}H_{\beta\delta}] = F_{\beta\alpha} - 2\mathbf{g}_{[\delta\alpha]}\mathbf{g}^{\delta\gamma}F_{\beta\gamma} \tag{15.6}$$

Using equations (15.6), (15.5), and (15.1), one gets

$$\begin{aligned} H_{\beta\alpha} &- (\eta^{\gamma\delta} - \eta^{\gamma\sigma}\eta^{\delta\tau}h_{\tau\sigma} + \eta^{\sigma\tau}\eta^{\gamma\rho}\eta^{\delta\varepsilon}h_{\sigma\rho}h_{\varepsilon\tau}) \cdot [h_{[\beta\delta]}H_{\gamma\alpha} + h_{[\gamma\alpha]}H_{\beta\delta}] \\ &= F_{\beta\alpha} - 2h_{[\delta\alpha]}F_{\beta\gamma}(\eta^{\delta\gamma} - \eta^{\delta\tau}\eta^{\gamma\sigma}h_{\sigma\tau} + \eta^{\delta\varepsilon}\eta^{\gamma\rho}h^{\sigma\tau}h_{\delta\varepsilon}h_{\rho\tau}) \end{aligned} \tag{15.7}$$

Let us expand $H_{\alpha\beta}$ in a power series in $h_{\alpha\beta}$:

$$H_{\alpha\beta} = \overset{(0)}{H}_{\alpha\beta} + \overset{(1)}{H}_{\alpha\beta} + \overset{(2)}{H}_{\alpha\beta} + \cdots \tag{15.8}$$

From (15.7) one gets

$$\overset{(0)}{H}_{\alpha\beta} = F_{\alpha\beta} \tag{15.9}$$

$$\overset{(1)}{H}_{\beta\alpha} = \eta^{\gamma\delta}(h_{[\beta\delta]}F_{\gamma\alpha} - h_{[\alpha\delta]}F_{\gamma\beta}) \tag{15.10}$$

$$\overset{(2)}{H}_{\beta\alpha} = \eta^{\gamma\delta}\eta^{\rho\sigma}[h_{(\rho\gamma)}(h_{[\alpha\delta]}F_{\alpha\beta} - h_{[\beta\delta]}F_{\sigma\alpha})] \tag{15.11}$$

Thus, we get for $H_{\alpha\beta}$ up to the second order in h

$$H_{\beta\alpha} = F_{\beta\alpha} + (\eta^{\gamma\delta} - h^{(\gamma\delta)})(h_{[\beta\delta]}F_{\gamma\alpha} - h_{[\alpha\delta]}F_{\gamma\beta}) \tag{15.12}$$

where

$$h^{(\gamma\delta)} = \eta^{\alpha\delta}\eta^{\rho\gamma}h_{(\rho\sigma)} \tag{15.13}$$

Let us pass to the electromagnetic Lagrangian in the nonsymmetric Kaluza–Klein (Jordan–Thiry) theory

$$\begin{aligned}\mathscr{L}_{\text{em}} &= \frac{1}{8\pi}[2(\mathbf{g}^{[\mu\nu]}F_{\mu\nu})^2 - H^{\mu\alpha}F_{\mu\alpha}] \\ &= \frac{1}{8\pi}[2(\mathbf{g}^{[\mu\nu]}F_{\mu\nu})^2 - \mathbf{g}^{\beta\mu}\mathbf{g}^{\gamma\alpha}H_{\beta\gamma}F_{\mu\alpha}]\end{aligned} \tag{15.14}$$

Let us expand $\mathscr{L}_{\text{em}}$ in a power series with respect to $h_{\alpha\beta}$,

$$\mathscr{L}_{\text{em}} = \overset{(0)}{\mathscr{L}}_{\text{em}} + \overset{(1)}{\mathscr{L}}_{\text{em}} + \overset{(2)}{\mathscr{L}}_{\text{em}} + \cdots \tag{15.15}$$

One gets from equations (15.14), (15.12), and (15.5)

$$\overset{(0)}{\mathscr{L}}_{\text{em}} = -\frac{1}{8\pi}\eta^{\beta\mu}\eta^{\gamma\alpha}F_{\beta\gamma}F_{\mu\alpha} \tag{15.16}$$

$$\overset{(1)}{\mathscr{L}}_{\text{em}} = \frac{1}{8\pi}[(\eta^{\beta\sigma}\eta^{\mu\tau} + \eta^{\beta\tau}\eta^{\mu\sigma})\eta^{\gamma\alpha}]h_{\tau\sigma}F_{\beta\gamma}F_{\mu\alpha} \tag{15.17}$$

$$\begin{aligned}\overset{(2)}{\mathscr{L}}_{\text{em}} = -\frac{1}{8\pi}[&\eta^{\mu\tau}(2\eta^{\beta\varepsilon}\eta^{\alpha\delta}\eta^{\gamma\alpha} + \eta^{\beta\sigma}\eta^{\gamma\varepsilon}\eta^{\alpha\delta} + \eta^{\varepsilon\sigma}\eta^{\gamma\alpha}\eta^{\beta\delta} - \eta^{\delta\sigma}\eta^{\gamma\alpha}\eta^{\beta\varepsilon} \\ &+ \eta^{\gamma\sigma}\eta^{\varepsilon\alpha}\eta^{\beta\delta} - \eta^{\gamma\sigma}\eta^{\delta\alpha}\eta^{\beta\varepsilon} - \tfrac{1}{2}\eta^{\alpha\sigma}\eta^{\beta\varepsilon}\eta^{\gamma\delta} + \tfrac{1}{2}\eta^{\alpha\sigma}\eta^{\beta\delta}\eta^{\gamma\varepsilon}) \\ &+ \tfrac{1}{2}\eta^{\gamma\alpha}(\eta^{\varepsilon\mu}\eta^{\delta\sigma}\eta^{\beta\tau} + \eta^{\varepsilon\mu}\eta^{\delta\tau}\eta^{\beta\sigma} - \eta^{\varepsilon\sigma}\eta^{\beta\tau}\eta^{\delta\mu} - \eta^{\delta\mu}\eta^{\varepsilon\tau}\eta^{\beta\sigma}) \\ &+ \tfrac{1}{2}\eta^{\beta\varepsilon}\eta^{\gamma\delta}(\eta^{\mu\sigma}\eta^{\alpha\tau} - \eta^{\mu\tau}\eta^{\alpha\sigma})]h_{\delta\varepsilon}h_{\tau\sigma}F_{\beta\gamma}F_{\mu\alpha}\end{aligned} \tag{15.18}$$

In the first order of approximation in $h_{\mu\nu} = \mathbf{g}_{\mu\nu} - \eta_{\mu\nu}$ one gets

$$\mathscr{L}_{\text{em}} = -\frac{1}{8\pi}[\eta^{\beta\mu}\eta^{\gamma\alpha}F_{\beta\gamma}F_{\mu\alpha} - (\eta^{\beta\sigma}\eta^{\mu\tau} + \eta^{\beta\tau}\eta^{\mu\sigma})\eta^{\gamma\alpha}h_{\tau\sigma}\cdot F_{\beta\gamma}F_{\mu\alpha}] \tag{15.19}$$

$$\mathscr{L}_{\text{em}} = -\frac{1}{8\pi}\mathbf{g}^{(\beta\mu)}\mathbf{g}^{(\gamma\alpha)}F_{\beta\gamma}F_{\mu\alpha} \quad \text{(in the first order for } h_{\mu\nu}\text{)} \tag{15.19a}$$

Thus, one easily notices that there are no skewon–photon terms up to the first order of approximation in $h_{\mu\nu}$. The skewon field $h_{[\mu\nu]}$ couples to the electromagnetic field from the second order of approximation. Thus, skewon–phonon interactions are negligible in the linear approximation. Let us pass to the Lagrangian for the scalar field in our theory,

$$\mathscr{L}_{\text{scal}}(\Psi)=\mathbf{g}^{[\nu\mu]}\mathbf{g}_{\delta\nu}\tilde{\mathbf{g}}^{(\alpha\delta)}\Psi_{,\mu}\Psi_{,\alpha} \tag{15.20}$$

The field Ψ is of course uncharged.

Let us expand $\mathscr{L}_{\text{scal}}(\Psi)$ into a power series with respect to $h_{\alpha\beta}$ i.e.,

$$\mathscr{L}_{\text{scal}}=\overset{(0)}{\mathscr{L}}_{\text{scal}}+\overset{(1)}{\mathscr{L}}_{\text{scal}}+\overset{(2)}{\mathscr{L}}_{\text{scal}}+\cdots \tag{15.21}$$

We get

$$\overset{(0)}{\mathscr{L}}_{\text{scal}}=0 \tag{15.22}$$

$$\overset{(1)}{\mathscr{L}}=0 \tag{15.23}$$

$$\overset{(2)}{\mathscr{L}}_{\text{scal}}=\eta^{\alpha\nu}\eta^{\mu\beta}\eta^{\rho\gamma}h_{[\alpha\beta]}h_{[\gamma\nu]}\Psi_{,\mu}\Psi_{,\rho} \tag{15.24}$$

It is easy to see that the field Ψ does not propagate if $h_{[\alpha\beta]}=0$. Thus, the propagation of the field Ψ is a purely nonsymmetric effect. This means that the "gravitational constant" is really constant if the skewon field vanishes. Let us suppose that the field Ψ is weak. This means that

$$|\Psi|\ll 1 \tag{15.25}$$

Thus, we expand around $\Psi=0$ (i.e., around the nonsymmetric Kaluza–Klein theory). We easily get

$$e^{-3\Psi}=1-3\Psi+\tfrac{9}{2}\Psi^2+\cdots \tag{15.26}$$

The field Ψ is the scalar field connected to the gravitational constant. The field Ψ is the scalar part of the gravitational field. Our approximation presented here is up to second order with respect to the gravitational field, i.e., with respect to $h_{\mu\nu}=\mathbf{g}_{\mu\nu}-\eta_{\mu\nu}$ and Ψ. In this way one easily gets for the Lagrangian in the nonsymmetric Jordan–Thiry theory (apart from the Lagrangian of the pure gravitational field from Moffat's theory of gravitation)

$$\mathscr{L}=(\overset{(0)}{\mathscr{L}}_{\text{em}}+\overset{(1)}{\mathscr{L}}_{\text{em}}+\overset{(2)}{\mathscr{L}}_{\text{em}})+\overset{(2)}{\mathscr{L}}_{\text{scal}}-3\Psi(\overset{(0)}{\mathscr{L}}_{\text{em}}+\overset{(1)}{\mathscr{L}}_{\text{em}})+\tfrac{9}{2}\Psi^2\overset{(0)}{\mathscr{L}}_{\text{em}} \tag{15.27}$$

It is easy to see that we get in this approximation the pseudo-mass-like term for the field Ψ

$$(\tfrac{9}{2}\overset{(0)}{\mathscr{L}}_{\text{cm}})\Psi^2 \tag{15.28}$$

and an interaction term

$$-[3(\overset{(0)}{\mathscr{L}}_{\rm em}+\overset{(1)}{\mathscr{L}}_{\rm em})]\Psi \tag{15.29}$$

The last expression (15.29) could be treated as an interaction of the field Ψ with sources, i.e.,

$$\Psi\cdot J \tag{15.30}$$

where

$$J=-3(\overset{(0)}{\mathscr{L}}_{\rm em}+\overset{(1)}{\mathscr{L}}_{\rm em}) \tag{15.31}$$

is a external source for Ψ. In the first-order of approximation in $h_{\mu\nu}=g_{\mu\nu}-\eta_{\mu\nu}$ and Ψ, one gets

$$\begin{aligned}\mathscr{L}=-\frac{1}{8\pi}[\eta^{\beta\alpha}\eta^{\gamma\mu}F_{\beta\gamma}F_{\mu\alpha}-(\eta^{\beta\sigma}\eta^{\mu\tau}+\eta^{\beta\tau}\eta^{\mu\sigma})\eta^{\gamma\alpha}h_{\tau\sigma}F_{\beta\gamma}F_{\mu\alpha}]\\ -3\Psi(\eta^{\beta\mu}\eta^{\gamma\alpha}F_{\beta\gamma}F_{\mu\alpha})\end{aligned} \tag{15.32}$$

i.e., we get an interaction term for the field Ψ,

$$[3(\eta^{\beta\mu}\eta^{\gamma\alpha}F_{\beta\gamma}F_{\mu\alpha})]\Psi \tag{15.33}$$

The field Ψ interacts, due to the term (15.30), with the electromagnetic field. Despite this, the field Ψ is uncharged. The propagator of the field Ψ vanishes in zeroth order of approximation with respect to $h_{\mu\nu}=g_{\mu\nu}-\eta_{\mu\nu}$. It vanishes also if $h_{[\mu\nu]}=0$. In the second order of approximation this propagator depends on the field $h_{[\mu\nu]}$. The exact forms of $\mathscr{L}^{(0)}_{\rm em}$, $\mathscr{L}^{(1)}_{\rm em}$, and $\mathscr{L}^{(2)}_{\rm em}$ are given by equations (15.16)–(15.18).

Let us remark on the convergence of the series appearing here. They converge for a sufficiently small $h_{\mu\nu}$. However, all the functions of $h_{\mu\nu}$ considered here (i.e., $H_{\mu\nu}$, $g^{\mu\nu}$, $\mathscr{L}_{\rm em}$) are well defined for any $h_{\mu\nu}$. They are rational functions of this variable. Moreover, the exact form of all these functions are hard to get.

16. EQUATIONS OF MOTION FOR A TEST PARTICLE IN THE LINEAR APPROXIMATION

Let us consider the equations for a test particle in the nonsymmetric Kaluza–Klein theory for $\rho=1$ or $\Psi=0$,[18,19,25]

$$\frac{\bar{D}u^\alpha}{d\tau}+\frac{q}{m_0}g^{\alpha\gamma}F_{\gamma\beta}u^\beta-\frac{q}{m_0}g^{[\alpha\gamma]}H_{\gamma\beta}u^\beta=0 \tag{16.1}$$

where $\bar{D}/d\tau$ is the covariant derivative along the line with respect to the connection $\bar{\omega}^{\alpha}_{\beta}$, q is the charge, and m_0 is the mass of a test particle.

Using equation (15.5), we easily write (16.1) in the second order of expansion with respect to $h_{\mu\nu}$ and we get

$$\frac{\bar{D}u^{\alpha}}{d\tau}+\frac{q}{m_0}(\eta^{\alpha\gamma}-\eta^{\alpha\sigma}\eta^{\gamma\tau}h_{\sigma\tau}+\eta^{\alpha\rho}\eta^{\gamma\sigma}\eta^{\tau\varepsilon}h_{\varepsilon\rho}h_{\sigma\tau})F_{\gamma\beta}u^{\beta}$$
$$+\frac{q}{m_0}(-h^{[\alpha\gamma]}+\eta^{[\alpha|\rho|}\eta^{\gamma]\sigma}\eta^{\tau\varepsilon}h_{\varepsilon\rho}h_{\rho\tau})$$
$$\times[F_{\gamma\beta}+(\eta^{\alpha\delta}-h^{(\sigma\delta)})(h_{[\gamma\delta]}F_{\sigma\beta}-h_{[\beta\delta]}F_{\sigma\gamma})]u^{\beta}=0 \qquad (16.2)$$

If Ψ (or ρ) is not constant (the general case), we find in terms of Ψ

$$\frac{\bar{D}u^{\alpha}}{d\tau}+\frac{q}{m_0}(\eta^{\alpha\gamma}-\eta^{\alpha\sigma}\eta^{\gamma\tau}h_{\tau\sigma}+\eta^{\alpha\rho}\eta^{\gamma\sigma}\eta^{\tau\varepsilon}h_{\varepsilon\rho}h_{\sigma\tau})F_{\gamma\beta}u^{\beta}$$
$$+\frac{q}{m_0}(-h^{[\alpha\gamma]}+\eta^{r[\alpha}\eta^{\gamma]\sigma}\eta^{\tau\varepsilon}h_{\varepsilon\rho}h_{\sigma\tau})$$
$$\times[F_{\gamma\beta}+(\eta^{\alpha\delta}-h^{(\alpha\delta)})(h_{[\gamma\delta]}F_{\sigma\beta}-h_{[\beta\delta]}F_{\sigma\gamma})]u^{\beta}$$
$$-\frac{1}{8}\left(\frac{q}{m_0}\right)^2(\Psi^2)_{,\beta}(\eta^{\beta\alpha}-\eta^{\alpha(\mu}\eta^{\nu)\beta}h_{\mu\nu})$$
$$-\frac{1}{4}\left(\frac{q}{m_0}\right)^2\Psi_{,\beta}(\eta^{\beta\alpha}-\eta^{\alpha(\mu}h^{\nu)\beta}h_{\nu\mu}+\eta^{\gamma(\alpha}\eta^{\beta)\sigma}\eta_{\nu\mu}h_{\mu\gamma}h_{\sigma\nu})=0 \qquad (16.3)$$

$$h^{[\alpha\beta]}=\eta^{\mu[\beta}\eta^{\alpha]\nu}h_{\mu\nu} \qquad (16.4)$$

It is easy to see that the skewon field has an influence on the motion of a test particle to first order in the expansion $h_{\mu}=\mathbf{g}_{\mu\nu}-\eta_{\mu\nu}$ and the scalar field Ψ.

17. THE GEODETIC EQUATIONS IN THE GENERAL CASE AND THE GEODETIC DEVIATION EQUATION

Let us consider the geodesic equation on P in the nonsymmetric Jordan–Thiry theory, which becomes an equation of motion of a test particle on E after taking a local section of P,

$$\frac{\tilde{D}u^{\alpha}}{d\tau}+\frac{q}{m_0}(\mathbf{g}^{\alpha\mu}F_{\mu\beta}-\mathbf{g}^{[\alpha\mu]}H_{\mu\beta})u^{\beta}-\frac{1}{8}\left(\frac{q}{m_0}\right)^2\tilde{g}^{(\beta\alpha)}\left(\frac{1}{\rho^2}\right)_{,\beta}=0 \qquad (17.1)$$

For $\tilde{\omega}^{\alpha}{}_{\beta}$, $F_{\mu\nu}$, $H_{\mu\nu}$, and ρ well defined on E, it has the same form as before. Equation (17.1) becomes equation (5.7) if $\rho=1$.

Let us find the physical interpretation of the additional term

$$-\frac{1}{8}\left(\frac{q}{m_0}\right)^2 \tilde{g}^{(\beta\alpha)}\left(\frac{1}{\rho^2}\right)_{,\beta} \tag{17.2}$$

This term describes the scalar, velocity-independent force acting on the test particle. The force depends on the "chemical composition" of the particle, because it has in front a factor $(q/m_0)^2$. Thus, it could be considered as a new type of the force, maybe the "fifth force."[(54,55)] In order to examine this, let us suppose that $F_{\mu\nu}=0$ $(H_{\mu\nu}=0)$ and consider the following equation:

$$\frac{\bar{D}u^\alpha}{d\tau}-\frac{1}{8}\left(\frac{q}{m_0}\right)^2 \tilde{g}^{(\alpha\beta)}\left(\frac{1}{\rho^2}\right)_{,\beta}=0 \tag{17.1a}$$

Let us multiply both sides of (17.1) by $g_{(\alpha\gamma)}u$ in order to understand the effect of an action of the scalar force on the test particle motion. One easily finds

$$\frac{d}{d\tau}\left(\frac{m_0}{2}g_{(\alpha\beta)}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\right)=\frac{1}{16}\frac{q^2}{m_0}\frac{d}{d\tau}\left(\frac{1}{\rho^2}\right) \tag{17.3}$$

where

$$u^\alpha=\frac{dx^\alpha}{d\tau} \quad \text{and} \quad \frac{d}{d\tau}\left(\frac{1}{\rho^2}\right)=\left(\frac{1}{\rho^2}\right)_{,\beta}\frac{dx^\beta}{d\tau}$$

It is well known that

$$m_0 g_{(\alpha\beta)}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}=E_p \tag{17.4}$$

has an interpretation as the total energy of a test particle in a rest frame. Thus, the scalar force changes the energy of a test particle in the following way:

$$\frac{dE_p}{d\tau}=\frac{q^2}{8m_0}\frac{d}{d\tau}\left(\frac{1}{\rho^2}\right) \tag{17.5}$$

Equation (17.5) goes to the first integral of motion

$$E_p-\frac{q^2}{8m_0\rho^2}=\text{const} \tag{17.6}$$

$$g_{(\alpha\beta)}u^\alpha u^\beta-\frac{q^2}{8m_0\rho^2}=\text{const} \tag{17.6a}$$

Thus, if the test particle is taken as a model of a charged planet in the solar system, the rest mass of a single planet will change during its motion according to equation (17.6). This result is easily understandable because of the physical interpretation of the field ρ. This field is connected to the effective gravitational constant

$$G_{\rm eff}=G_N\rho^2 \tag{17.7}$$

(G_N is Newton's constant).

Thus, if $\rho\neq$const, the effective strength of the gravitational interaction changes during the motion and the energy of a test particle changes. Moreover, the total energy of a test particle and the field ρ is constant. Such a change must be secular if considered in the solar system and this is the case, because the field ρ seems to be massive with a short range.

In general the scalar force can act as a "friction" or an "amplification" force. If such an effect exists in the solar system, it must be connected to the global (cosmological) change of the effective gravitational constant coming from the cosmological solution in the nonsymmetric Jordan–Thiry theory. Unfortunately, such a solution is unknown.

For this it seems that the $\rho=1$ case is quite important. It corresponds to the Ansatz $\gamma_{55}(x)=$const and leads to the normalization of the four-velocity during the motion of a test particle,

$$g_{\alpha\beta}u^\alpha(\tau)u^\beta(\tau)=\text{const} \tag{17.8}$$

The general five-dimensional case does not preserve this condition. Thus, if we demand the condition (17.8) in this nonsymmetric Kaluza–Klein theory, $\gamma_{55}(x)=$const is not an Ansatz, but rather a conclusion from (17.8). In the next section we deal with this case in more detail, including field equations and their properties.

Let us consider the geodetic deviation equation in our theory,

$$u^B\nabla_B v^A-[\nabla_M,\nabla_B]u^A u^B\zeta^M=0 \tag{17.9}$$

or

$$u^B\nabla_B v^A+R^A{}_{CMB}u^C\zeta^M u^B-Q^N{}_{MB}\nabla_N u^A\zeta^M u^B=0 \tag{17.9*}$$

In GRT one has

$$\frac{D^2\zeta^\alpha}{d\tau^2}+\tilde{\bar{R}}^\alpha{}_{\beta\gamma\delta}\frac{dx^\beta}{d\tau}\zeta^\gamma\frac{dx^\delta}{d\tau}=0 \tag{17.10}$$

In the presence of nonzero torsion one gets

$$u^\beta\bar\nabla_\beta u^\alpha+\bar R^\alpha{}_{\beta\gamma\delta}u^\beta\zeta^\gamma u^\delta-\bar Q^\nu{}_{\mu\beta}\bar\nabla_\nu u^\alpha\zeta^\mu u^\beta=0 \tag{17.11}$$

where $u^\alpha = dx^\alpha/d\tau$ and $v^\alpha = d\zeta^\alpha/d\tau$. We suppose, of course, as usual

$$u^B \nabla_B u^A = 0 \tag{17.9a}$$

In GRT we have

$$\frac{\tilde{\bar{D}} u^\alpha}{d\tau} = 0 \tag{17.10a}$$

and in the presence of torsion

$$u^\beta \bar{\nabla}_\beta u^\alpha = 0 \tag{17.11a}$$

In this way we consider a generalization of the geodetic deviation equation to the five-dimensional case and in a non-Riemannian geometry. In GRT, $\zeta^\alpha(\tau)$ is called the geodetic deviation vector and equations (17.10) and (17.10a) give a physical interpretation for the curvature tensor. Using equations (4.8), (6.9a)–(6.9h), and (6.1)–(6.4), one derives from equation (17.9*).

$$\begin{aligned}
&(u^\beta \tilde{\bar{\nabla}}_\beta v^\alpha + \bar{R}^\alpha{}_{\beta\mu\nu} u^\beta \zeta^\mu u^\nu - \bar{Q}^\nu{}_{\mu\beta}(\bar{\Gamma}) \bar{\nabla}_\nu u^\alpha \zeta^\mu u^\beta) + \rho^2 v^5 \mathbf{g}^{\alpha\beta}(H_{\gamma\beta} - 2F_{\gamma\beta}) u^\gamma \\
&+ \frac{1}{2}\frac{q}{m_0} \mathbf{g}^{\delta\alpha} H_{\delta\gamma} v^\gamma + \frac{1}{2\rho} v^5 \frac{q}{m_0} \mathbf{g}^{(\gamma\alpha)} \rho_{,\gamma} + 2\rho^2 \mathbf{g}^{\alpha\beta}[\mathbf{g}^{\delta\alpha} H_{\delta\beta} F^{\mu\nu} \\
&- \mathbf{g}^{\alpha\gamma}(H_{[\nu|\gamma|} - 2F_{[\nu|\gamma|}) \cdot H_{|\beta|\mu]}] \cdot u^\beta \zeta^\mu u^\nu - \zeta^5[\bar{\nabla}_\mu(\rho^2 \mathbf{g}^{\delta\alpha} H_{\delta\beta}) + \rho H_{\beta\mu} \tilde{\mathbf{g}}^{(\gamma\alpha)} \rho_{,\gamma} \\
&+ \rho \mathbf{g}^{\alpha\nu}(H_{\mu\nu} - 2F_{\mu\nu}) \mathbf{g}_{\beta\delta} \tilde{\mathbf{g}}^{(\gamma\delta)} \rho_{,\gamma}] \cdot u^\beta u^\mu + \frac{q}{2\rho m_0}\{2\bar{\nabla}_{[\mu}[\rho^2 \mathbf{g}^{\alpha\beta}(H_{\nu]\beta} - 2F_{\nu]\beta}) \\
&+ \rho^2 \mathbf{g}^{\alpha\beta}(H_{\gamma\beta} - 2F_{\gamma\beta}) \bar{Q}^\gamma{}_{\mu\nu}(\bar{\Gamma}) + 2\rho^2 \tilde{\mathbf{g}}^{(\gamma\alpha)} \rho_{,\gamma} F_{\mu\nu} \\
&+ 2\rho \mathbf{g}^{\alpha\beta} \mathbf{g}_{\delta[\nu} \tilde{\mathbf{g}}^{(\alpha\delta)} \rho_{,|\alpha|}(H_{\mu]\beta} - 2F_{\mu]\beta}) + \bar{\nabla}_\mu(\rho^2 \mathbf{g}^{\delta\alpha} H_{\delta\nu}) + \rho H_{\mu\nu} \tilde{\mathbf{g}}^{(\gamma\alpha)} \rho_{,\gamma} \\
&+ \rho \mathbf{g}^{\alpha\beta}(H_{\mu\beta} - 2F_{\mu\beta}) \mathbf{g}_{\nu\delta} \tilde{\mathbf{g}}^{(\alpha\delta)} \rho_{,\gamma}]\} u^\nu \zeta^\mu + \frac{q}{4\rho^4 m_0}[\bar{\nabla}_\mu(2\rho \tilde{\mathbf{g}}^{(\gamma\alpha)} \rho_{,\gamma}) \\
&+ \rho^4 \mathbf{g}^{\delta\alpha} \mathbf{g}^{\gamma\beta} H_{\delta\gamma}(H_{\beta\mu} - 2F_{\beta\mu}) - \mathbf{g}_{\delta\mu} \tilde{\mathbf{g}}^{(\gamma\alpha)} \tilde{\mathbf{g}}^{(\nu\delta)} \rho_{,\gamma} \rho_{,\nu}]\left(\frac{q}{m_0}\zeta^\mu - 2\rho^2 \zeta^5 u^\mu\right) \\
&+ \frac{q}{2m_0} \bar{Q}^\nu{}_{\mu\beta}(\bar{\Gamma}) \zeta^\mu u^\beta \mathbf{g}^{\alpha\gamma}(H_{\nu\gamma} - 2F_{\nu\gamma}) \\
&- (\mathbf{g}^{(\delta\nu)} H_{\delta\mu} - \mathbf{g}^{\nu\beta} F_{\delta\mu}) \bar{\nabla}_\nu u^\alpha \left(\zeta^\mu \frac{q}{m_0} - 2\rho^2 \zeta^5 u^\mu\right) \\
&+ \frac{1}{2}\frac{q}{m_0}(\mathbf{g}^{(\delta\nu)} H_{\delta\mu} - \mathbf{g}^{\nu\delta} F_{\delta\mu}) \mathbf{g}^{\alpha\beta}(H_{\nu\beta} - 2F_{\nu\beta}) \zeta^\mu \frac{q}{m_0} - 2\rho^2 \zeta^5 u^\mu
\end{aligned}$$

$$+2\rho^2(F_{\mu\beta}-H_{\mu\beta})\zeta^\mu u^\beta \mathbf{g}^{\delta\alpha}H_{\delta\gamma}u^\gamma-\frac{q}{m_0\rho}(F_{\mu\beta}-H_{\mu\beta})\tilde{\mathbf{g}}^{(\gamma\alpha)}\rho_{,\gamma}$$

$$-\frac{1}{\rho^3}\mathbf{g}_{[\beta\gamma]}\tilde{\mathbf{g}}^{(\mu\delta)}\rho_{,\mu}\left(\rho^2\mathbf{g}^{\gamma\alpha}H_{\gamma\beta}u^\beta+\frac{q}{2\rho m_0}\tilde{\mathbf{g}}^{(\gamma\alpha)}\rho_{,\gamma}\right)$$

$$+\left(2\rho^2\zeta^5u^\mu-\frac{q}{m_0}\zeta^\mu\right)=0 \tag{17.12}$$

and

$$\frac{dv^5}{d\tau}+H_{\gamma\beta}v^\gamma u^\beta+\frac{v^5}{\rho}\mathbf{g}_{\delta\beta}\tilde{\mathbf{g}}^{(\alpha\beta)}\rho_{,\alpha}u^\beta+\frac{q}{2\rho^3m_0}\mathbf{g}_{\gamma\beta}\tilde{\mathbf{g}}^{(\alpha\beta)}\rho_{,\alpha}v^\gamma$$

$$+\Bigg(2\bar{\nabla}_{[\mu}H_{\nu]\beta}+H_{\gamma\beta}\bar{Q}^\gamma{}_{\mu\nu}(\bar{\Gamma})+\frac{2}{\rho}\mathbf{g}_{\beta\delta}\tilde{\mathbf{g}}^{(\alpha\delta)}\rho_{,\alpha}F_{\mu\nu}$$

$$+\frac{2}{\rho}\mathbf{g}_{\delta[\mu}\mathbf{g}^{(\alpha\delta)}\rho_{,|\alpha|}H_{|\beta|\nu]}\Bigg)u^\beta\zeta^\mu u^\nu+\zeta^5\Bigg[\bar{\nabla}_\mu\left(\frac{2}{\rho}\mathbf{g}_{\beta\delta}\tilde{\mathbf{g}}^{(\alpha\delta)}\rho_{,\alpha}\right)+\rho^2\mathbf{g}^{\delta\gamma}H_{\delta\beta}H_{\gamma\mu}$$

$$+\frac{1}{\rho^2}\mathbf{g}_{\delta\mu}\mathbf{g}_{\beta\gamma}\tilde{\mathbf{g}}^{(\alpha\delta)}\tilde{\mathbf{g}}^{(\nu\gamma)}\rho_{,\alpha}\rho_{,\nu}\Bigg]u^\beta u^\mu+\frac{q}{2\rho^2m_0}\Bigg[2\bar{\nabla}_{[\mu}\left(\frac{1}{\rho}\mathbf{g}_{|\delta|\nu]}\tilde{\mathbf{g}}^{(\alpha\delta)}\rho_{,\alpha}\right)$$

$$+\frac{1}{\rho}\mathbf{g}_{\delta\gamma}\tilde{\mathbf{g}}^{(\alpha\delta)}\rho_{,\alpha}\bar{Q}^\gamma{}_{\mu\nu}(\bar{\Gamma})-2\rho^2\mathbf{g}^{\beta\alpha}H_{\beta[\nu}(H_{\mu]\alpha}-2F_{\mu]\alpha})$$

$$+\frac{2}{\rho^2}\mathbf{g}_{\delta[\mu}\mathbf{g}_{|\gamma|\nu]}\tilde{\mathbf{g}}^{(\alpha\delta)}\tilde{\mathbf{g}}^{(\beta\gamma)}\rho_{,\alpha}\rho_{,\beta}+\bar{\nabla}_\mu\left(\frac{2}{\rho}\mathbf{g}_{\nu\delta}\tilde{\mathbf{g}}^{(\alpha\delta)}\rho_{,\alpha}\right)+\rho^2\mathbf{g}^{\delta\gamma}H_{\delta\nu}H_{\gamma\mu}$$

$$+\frac{1}{\rho^2}\mathbf{g}_{\delta\mu}\mathbf{g}_{\nu\gamma}\tilde{\mathbf{g}}^{(\alpha\delta)}\tilde{\mathbf{g}}^{(\beta\gamma)}\rho_{,\alpha}\rho_{,\beta}\Bigg]\zeta^\mu u^\nu$$

$$+\frac{q}{4\rho^4m_0}[\rho\tilde{\mathbf{g}}^{(\gamma\beta)}\rho_{,\gamma}H_{\beta\mu}-\rho\tilde{\mathbf{g}}^{(\alpha\delta)}\rho_{,\alpha}(H_{\mu\delta}-2F_{\mu\delta})]\left(2\rho^2\zeta^5u^\mu-\frac{q}{m_0}\zeta^\mu\right)$$

$$-\bar{Q}^\nu{}_{\mu\beta}(\bar{\Gamma})\zeta^\mu u^\beta\left(H_{\gamma\nu}u^\gamma-\frac{\rho_{,\nu}}{\rho^3}\frac{q}{m_0}+\frac{q}{2\rho^3m_0}\mathbf{g}_{\delta\nu}\tilde{\mathbf{g}}^{(\alpha\delta)}\rho_{,\alpha}\right)$$

$$-2\rho^2(\mathbf{g}^{(\alpha\delta)}H_{\delta\beta}-\mathbf{g}^{\alpha\delta}F_{\delta\beta})\left(H_{\gamma\nu}u^\gamma+\frac{q}{2\rho^3m_0}\mathbf{g}_{\gamma\nu}\tilde{\mathbf{g}}^{(\alpha\gamma)}\rho_{,\alpha}\right)\left(\zeta^5u^\beta-\frac{q}{2m_0\rho^2}\zeta^\mu\right)$$

$$-\frac{2}{\rho}(F_{\mu\beta}-H_{\mu\beta})u^{\beta}\mathbf{g}_{\gamma\delta}\tilde{\mathbf{g}}^{(\alpha\delta)}\rho_{,\alpha}\zeta^{\mu}u^{\gamma}-\frac{1}{\rho^{4}}\mathbf{g}_{[\beta\delta]}\tilde{\mathbf{g}}^{(\alpha\delta)}\rho_{,\alpha}\mathbf{g}_{\mu\nu}\tilde{\mathbf{g}}^{(\gamma\nu)}\rho_{,\gamma}u^{\mu}$$

$$\times\left(2\rho\zeta^{5}u^{\beta}-\zeta^{\beta}\frac{q}{m_0}\right)=0 \qquad (17.13)$$

Let us consider a simpler case of equations (17.12)–(17.13), i.e., the classical Kaluza–Klein theory case. One gets from equations (17.12)–(17.13) for $F_{\mu\nu}=H_{\mu\nu}$ and $\rho=1$

$$(u^{\beta}\tilde{\tilde{\nabla}}_{\beta}v^{\alpha}+\tilde{\tilde{R}}{}^{\alpha}{}_{\beta\mu\nu}u^{\beta}\zeta^{\mu}u^{\nu})+v^{5}\mathbf{g}^{\alpha\beta}F_{\delta\beta}u^{\gamma}+\frac{1}{2}\frac{q}{m_0}\mathbf{g}^{\delta\alpha}F_{\delta\gamma}v^{\gamma}$$

$$+2(\mathbf{g}^{\delta\alpha}F_{\delta\beta}F_{\mu\nu}+\mathbf{g}^{\alpha\gamma}F_{[\nu|\gamma|}F_{|\beta|\mu]})u^{\beta}\zeta^{\mu}u^{\nu}-\zeta^{5}\tilde{\tilde{\nabla}}_{\mu}(\mathbf{g}_{\delta\alpha}F_{\delta\beta})u^{\beta}u^{\mu}$$

$$+\frac{1}{2}\frac{q}{m_0}[\tilde{\tilde{\nabla}}_{\mu}(\mathbf{g}^{\delta\alpha}F_{\delta\nu})-2\tilde{\tilde{\nabla}}_{[\mu}(\mathbf{g}^{\alpha\beta}F_{\nu]\beta})]u^{\nu}\zeta^{\mu}$$

$$-\frac{q}{4m_0}\mathbf{g}^{\delta\alpha}\mathbf{g}^{\gamma\beta}F_{\delta\gamma}F_{\beta\mu}\left(\frac{q}{m_0}\zeta^{\mu}-2\zeta^{5}u^{\mu}\right)=0 \qquad (17.14)$$

and

$$\frac{dv^{5}}{d\tau}+F_{\gamma\beta}v^{\gamma}u^{\beta}-\tilde{\tilde{\nabla}}_{\nu}F_{\mu\beta}u^{\nu}\zeta^{\mu}u^{\beta}+\zeta^{5}\mathbf{g}^{\delta\gamma}F_{\delta\beta}F_{\gamma\mu}u^{\beta}u^{\mu}+\frac{q}{2m_0}\mathbf{g}^{\delta\gamma}F_{\delta\nu}F_{\gamma\mu}\zeta^{\mu}u^{\nu}=0 \qquad (17.15)$$

where we use

$$u^{A}=(u^{\alpha},u^{5})=\left(u^{\alpha},\frac{q}{2\rho^{2}m_0}\right)$$

$$\zeta^{A}=(\zeta^{\alpha},\zeta^{5})$$

and

$$v^{A}=(v^{\alpha},v^{5})=\left(\frac{D\zeta^{\alpha}}{d\tau},\frac{d\zeta^{5}}{d\tau}\right)$$

We have as usual $q/m_0=\text{const}$ and for $u^{\alpha}=dx^{\alpha}/d\tau$ we have equation (17.1) or the simpler equation in the Riemannian case,

$$\frac{\tilde{\tilde{D}}u^{\alpha}}{d\tau}+\frac{q}{m_0}\mathbf{g}^{\alpha\mu}F_{\mu\beta}=0 \qquad (17.16)$$

Moreover, in the Riemannian case we have

$$u^5 = \frac{dx^5}{d\tau} = \frac{1}{2}\frac{q}{m_0} \tag{17.17}$$

or

$$x^5 = \frac{q}{2m_0}(\tau - \tau_0) + x_0^5 \tag{17.18}$$

Moreover, in this case we consider a flow of geodesics given by $x^A = x^A(\tau, \sigma)$ and

$$\zeta^A = \left(\frac{\partial x^\alpha}{\partial \sigma}, \frac{\partial x^5}{\partial \sigma}\right)\bigg|_{\sigma=\sigma_0}, \qquad \sigma, \sigma_0 \in U \subset R$$

σ is a parameter such that for every $\sigma_1 \neq \sigma_2$, $x^A(\tau, \sigma_1)$ and $x^A(\tau, \sigma_2)$ are different geodesics. One can say that we have a family of geodesic curves $\Gamma(\sigma)$, $\sigma \in U \subset R'$. The geodesic considered by us is $\Gamma(\sigma_0)$, i.e., for $\sigma = \sigma_0 \in U$. Thus,

$$\zeta^5 = \frac{1}{2}(\tau - \tau_0)\frac{d}{d\sigma}\left(\frac{q}{m_0}\sigma\right)\bigg|_{\sigma=\sigma_0} + \frac{dx_0^5}{d\sigma}\bigg|_{\sigma=\sigma_0} \tag{17.19}$$

$$v^5 = \frac{1}{2}\frac{d}{d\sigma}\left(\frac{q}{m_0}\sigma\right)\bigg|_{\sigma=\sigma_0} \tag{17.20}$$

$$\frac{dv^5}{d\tau} = 0 \tag{17.21}$$

In this way v^5 is an integral of motion and equation (17.15) is redundant. Equation (17.14) after the substitution of equations (17.19) and (17.20) represents together with equation (17.16) an analogue of the geodetic deviation equations for a charged particle equation of motion.

In the general case (i.e., non-Riemannian with $\rho \neq \text{const}$) the situation is more complex. Now we have

$$\frac{dx^5}{d\tau} = u^5 = \frac{1}{2\rho^2}\frac{q}{m_0} \tag{17.22}$$

where

$$\rho = \rho(\tau, \sigma) = \rho(x(\tau, \sigma))$$

and

$$x^5 = \frac{1}{2}\frac{q}{m_0}\int_{\tau_0}^{\tau}\frac{d\tau}{\rho^2(\tau,\sigma)} + x_0^5(\sigma) \tag{17.23}$$

$$\zeta^5 = \frac{1}{2}\frac{\partial}{\partial\sigma}\left(\frac{q}{m_0}\sigma\int_{\tau_0}^{\tau}\frac{d\tau}{\rho^2(\tau,\sigma)}\right|_{\sigma=\sigma_0} + \left.\frac{dx_0^5}{d\sigma}\right|_{\sigma=\sigma_0} \tag{17.24}$$

$$v^5 = \frac{d\zeta^5}{d\tau} = \frac{1}{2}\frac{\partial}{\partial\sigma}\left.\left(\frac{q}{m_0}\sigma\frac{1}{\zeta^2(\tau,\sigma)}\right)\right|_{\sigma=\sigma_0} \tag{17.25}$$

After substitution of equations (17.24)–(17.25) into equations (17.12)–(17.13) we get the geodetic deviation equations in nonsymmetric Jordan–Thiry theory (NJTT). They are analogous to the deviation equation for the charged particle equation of motion in NJTT.

Let us remark on a physical interpretation of the vector $\zeta^A = (\zeta^\alpha, \zeta^5)$. The vector ζ^A, "geodesic separation," is the displacement (tangent vector) from a point on the fiducial geodesic to a point on a nearby geodesic characterized by the same value of an affine parameter τ. Thus, $v^A = (v^\alpha, v^5)$ means a relative "velocity" and $u^B\nabla_B v^A$ a relative "acceleration." The relative "acceleration" equals, according to equation (17.9), a commutator of covariant derivatives. Thus, we get "tidal forces" in NJTT (five-dimensional case), i.e., for charged test particles. Equation (17.12) gives us "tidal forces" for charged test particles in NJTT. In this equation we get gravitational "tidal forces" from NGT, electromagnetic "tidal forces," and additional effects which can be treated as gravitoelectromagnetic tidal forces. The scalar field ρ is also a source of additional "tidal forces." These new effects are "interference effects" between gravitational and electromagnetic interactions described by NJTT. The commutator in equation (17.9) can be treated as a five-dimensional analogue of "tide-producing gravitational forces." In our case this is "tide-producing gravitoelectromagnetic forces." Our equations are defined on a bundle manifold $\underline{P}$. Due to the fact that $U(1)$ is Abelian, we get exactly the same equations.

Finally, let us remark that equation (17.12) represents tidal gravitoelectromagnetic forces and equation (17.13) is a new type of equation. It governs the relative change of $(q/m_0)\sigma$ for different test particles via v^5 [see equation (17.25)].

18. FIELD EQUATIONS FOR THE NONSYMMETRIC KALUZA–KLEIN THEORY (CASE WITH $\rho = 1$)

Let us consider a simpler version of our theory, i.e., the nonsymmetric Kaluza–Klein theory (NKKT). In this case $\rho = 1$ and from the Lagrangian of the theory [equations (6.5)] we get the field equations

$$\bar{R}_{\alpha\beta}(\bar{W}) - \tfrac{1}{2}\mathbf{g}_{\alpha\beta}\bar{R}(\bar{W}) = 8\pi \overset{\text{em}}{T}_{\alpha\beta} \tag{18.1}$$

$$\underset{\sim}{\mathbf{g}}^{[\mu\nu]}{}_{,\nu} = 0 \tag{18.2}$$

$$\mathbf{g}_{\mu\nu,\sigma} - \mathbf{g}_{\xi\nu}\bar{\Gamma}^{\xi}_{\mu\sigma} - \mathbf{g}_{\mu\xi}\bar{\Gamma}^{\xi}_{\sigma\nu} = 0 \tag{18.3}$$

$$\partial_\mu(\underset{\sim}{H}^{\alpha\mu}) = 2\underset{\sim}{\mathbf{g}}^{[\alpha\beta]}\partial_\beta(\mathbf{g}^{[\mu\nu]}F_{\mu\nu}) \tag{18.4}$$

We can rewrite equation (18.4)

$$\bar{\nabla}_\mu H^{\alpha\mu} = 2\mathbf{g}^{\alpha\beta}\bar{\nabla}_\beta(\mathbf{g}^{[\mu\nu]}F_{\mu\nu}) \tag{18.4a}$$

Recall that the current on the right-hand side of equation (18.4) has the property of the topological current, because

$$\partial_\alpha \underset{\sim}{J}^{\alpha} = \frac{1}{2\pi}\,\partial_\alpha[\underset{\sim}{\mathbf{g}}^{[\alpha\beta]}\partial_\beta(\mathbf{g}^{4[\mu\nu]}F_{\mu\nu})] = \frac{1}{2\pi}\,\underset{\sim}{\mathbf{g}}^{[\alpha\beta]}\partial_\alpha\,\partial_\beta(\mathbf{g}^{[\mu\nu]}F_{\mu\nu}) = 0 \tag{18.5}$$

modulo equation (18.2), i.e., it is conserved, by its definition.

We have

$$\overset{\text{em}}{T}_{\alpha\beta} = \frac{1}{4\pi}\,\{\mathbf{g}_{\gamma\beta}\mathbf{g}^{\tau\mu}\mathbf{g}^{\varepsilon\gamma}H_{\mu\alpha}H_{\tau\varepsilon} - 2\mathbf{g}^{[\mu\nu]}F_{\mu\nu}F_{\alpha\beta}$$
$$- \tfrac{1}{4}\mathbf{g}_{\alpha\beta}[H^{\mu\nu}F_{\mu\nu} - 2(\mathbf{g}^{[\mu\nu]}F_{\mu\nu})^2]\} \tag{18.6}$$

$$\underset{\sim}{\mathbf{g}}^{[\mu\nu]} = \sqrt{-\mathbf{g}}\;\mathbf{g}^{[\mu\nu]}$$
$$\underset{\sim}{H}^{\mu\alpha} = \sqrt{-\mathbf{g}}\;\mathbf{g}^{\beta\mu}\mathbf{g}^{\gamma\alpha}H_{\beta\gamma} \tag{18.7}$$

The tensor $H_{\mu\nu}$ has an interpretation as a second electromagnetic field strength tensor (see Section 8). We have

$$g^{\alpha\beta}\overset{\text{em}}{T}_{\alpha\beta} = 0 \tag{18.8}$$

Equations (18.1)–(18.4) can be written in the following form:

$$\bar{R}_{(\alpha\beta)}(\bar{\Gamma}) = 8\pi \overset{\text{em}}{T}_{(\alpha\beta)} \tag{18.9}$$

$$\bar{R}_{[[\alpha\beta],\gamma]}(\bar{\Gamma}) - 8\pi \overset{\text{em}}{T}_{[[\alpha\beta],\gamma]} = 0 \tag{18.10}$$

$$\bar{\Gamma}_\mu = 0 \tag{18.11}$$

$$\mathbf{g}_{\mu\nu,\sigma} - \mathbf{g}_{\xi\nu}\bar{\Gamma}^{\xi}{}_{\mu\sigma} - \mathbf{g}_{\mu\xi}\bar{\Gamma}^{\xi}{}_{\sigma\nu} = 0 \tag{18.12}$$

$$\partial_\mu(\underset{\sim}{H}^{\alpha\mu} - 2\underset{\sim}{\mathbf{g}}^{[\alpha\mu]}(\mathbf{g}^{[\nu\beta]}F_{[\nu\beta]})) = 0 \tag{18.13}$$

where $\bar{R}_{\alpha\beta}(\bar{\Gamma})$ is the Moffat–Ricci tensor for the connection

$$\bar{\omega}^{\alpha}{}_{\beta} = \bar{\Gamma}^{\alpha}{}_{\beta\gamma}\bar{\theta}^{\gamma}$$
$$\bar{\Gamma}_{\mu} = \bar{\Gamma}^{\alpha}_{[\mu\alpha]} \tag{18.14}$$

The condition (18.11) is equivalent to (18.2).

19. SPHERICALLY SYMMETRIC FIELDS IN THE NONSYMMETRIC KALUZA–KLEIN THEORY

Let us suppose that the fundamental fields in the nonsymmetric Kaluza–Klein theory possess spherical symmetry. According to Refs. 71 and 80–86, one gets

$$g_{\mu\nu} = \begin{bmatrix} -\alpha & 0 & 0 & \omega \\ 0 & -\beta & f\sin\theta & 0 \\ 0 & -f\sin\theta & -\beta\sin^2\theta & 0 \\ -\omega & 0 & 0 & \gamma \end{bmatrix} \tag{19.1}$$

where α, β, γ, f, and ω are real functions of r and t, with $\alpha, \gamma > 0$. In addition,

$$F_{14} = E(r, t), \qquad F_{23} = B\sin\theta \tag{19.2}$$

and all other components of $F_{\mu\nu}$ vanish. For $g^{\mu\nu}$, the only nonvanishing components are

$$g^{11} = \frac{\gamma}{\omega^2 - \alpha\gamma}$$
$$g^{22} = g^{33}\sin^2\theta = -\frac{\beta}{\beta^2 + f^2}$$
$$g^{44} = -\frac{\alpha}{\omega^2 - \alpha\gamma}$$
$$g^{[14]} = \frac{\omega}{\omega^2 - \alpha\gamma}$$
$$g^{[23]}\sin\theta = \frac{f}{\beta^2 + f^2} \tag{19.3}$$

We suppose that

$$\omega^2 - \alpha\gamma \neq 0 \qquad \text{and} \qquad \beta^2 + f^2 \neq 0 \tag{19.4}$$

Let us suppose that $H_{\alpha\beta}$ is also spherically symmetric, so that

$$H_{14}=D(r, t), \qquad H_{23}=H \sin \theta \tag{19.5}$$

and other components vanish. Using equations (4.9), (19.1), and (19.3) it can be shown that

$$H_{14}=F_{14}=E(r, t) \tag{19.6}$$

$$H_{23}=F_{23}=B \sin \theta$$

The Bianchi identity (4.*) yields

$$B=B_0=\text{const} \tag{19.7}$$

From equation (18.2) one gets

$$\frac{\omega^2}{\alpha\gamma-\omega^2}=\frac{l^2}{\beta^2+f^2} \tag{19.8}$$

where l^2 is a constant of integration. In Moffat's theory of gravitation this constant has an interpretation as a fermion charge. From equation (18.13) we have

$$\frac{E}{\omega}=\frac{-(Q/l^2)(\beta^2+f^2)+4fB_0}{\beta^2+f^2+4l^4} \tag{19.9}$$

where Q is an integration constant. In the intermediate stages of calculation we used the following expressions for $H^{\mu\alpha}$ and $\sqrt{-\mathbf{g}}$:

$$H^{14}=-\frac{H_{14}}{\alpha\gamma-\omega^2}=\frac{-E}{\alpha\gamma-\omega^2} \tag{19.10}$$

$$H^{23}=\frac{B_0}{\beta^2+f^2} \tag{19.11}$$

$$\sqrt{-\mathbf{g}}=\sin \theta[(\alpha\gamma-\omega^2)(\beta^2+f^2)]^{1/2} \tag{19.12}$$

Thus, we get equations (18.9)–(18.12) plus the algebraic relations (19.7)–(19.9). From equation (18.10) we get immediately

$$R_{[23]}(\bar{\Gamma})-8\pi \overset{\text{em}}{T}_{[23]}=C_1 \sin \theta \tag{19.13}$$

where $C_1 = \text{const}$ is an integration constant and

$$\frac{8\pi}{\sin\theta}\overset{\text{em}}{T}_{[23]} = -\frac{7fB_0^2}{B^2+f^2} + \frac{fl^4}{\beta^2+f^2}\left(\frac{E}{\omega}\right)^2 + 4f\left(\frac{fB_0}{\beta^2+f^2} - \frac{l^4}{\beta^2+f^2}\frac{E}{\omega}\right)^2 + \frac{8B_0 l^4}{\beta^2+f^2}\frac{E}{\omega} \tag{19.14}$$

Equations (18.11) and (18.12) were solved in Ref. 82, in which the Ricci tensor was written down for such a connection.

Note that the Moffat–Ricci tensor is a linear combination of the ordinary Ricci tensor and the second contraction of the curvature tensor. However, equations (18.2) and (18.3) imply that[34]

$$\bar{\bar{\Gamma}}^{\alpha}{}_{[\mu\alpha]} = 0 \tag{19.15}$$

and

$$\bar{\Gamma}^{\beta}_{\nu\beta} = [\ln((-\mathrm{g})^{1/2})]_{,\nu} \tag{19.16}$$

so that the second contraction is given by

$$\bar{R}^{\alpha}_{\alpha\mu\nu} = \tfrac{1}{2}(\bar{\Gamma}^{\beta}_{(\mu\beta),\nu} - \bar{\Gamma}^{\beta}_{(\nu\beta),\mu}) = 0 \tag{19.17}$$

Consequently the Moffat–Ricci tensor in this case is identically equal to the ordinary Ricci tensor used by Pant,[82] which we shall denote by $A_{\mu\nu}(\bar{\Gamma})$.

Thus, we get the following equations:

$$\begin{aligned} A_{(\mu\nu)}(\bar{\Gamma}) &= 8\pi\overset{\text{em}}{T}_{(\mu\nu)} \\ A_{[23]}(\bar{\Gamma}) - 8\pi\overset{\text{em}}{T}_{[23]} &= C_1 \sin\theta \end{aligned} \tag{19.18}$$

where

$$8\pi\overset{\text{em}}{T}_{11} = \alpha\frac{l^4}{\beta^2+f^2}\frac{E^2}{\omega^2} + \frac{\alpha B_0{}^2}{\beta^2+f^2} - 4\alpha\left(\frac{fB_0}{\beta^2+f^2} - \frac{l^4}{\beta^2+f^2}\frac{E}{\omega}\right)^2 \tag{19.19}$$

Using equation (19.9), we can write the last term in equation (19.19) in the form

$$-4\alpha\left(\frac{fB_0+Ql^2}{\beta^2+f^2+4l^4}\right)^2 \tag{19.20}$$

Moreover, it can be shown that

$$8\pi \overset{\text{em}}{T}_{44}=-\frac{\gamma}{\alpha}\,8\pi \overset{\text{em}}{T}_{11} \tag{19.21}$$

$$\begin{aligned}8\pi \overset{\text{em}}{T}_{22}&=\frac{8\pi}{\sin^2\theta}\overset{\text{em}}{T}_{33}=\frac{\beta}{\alpha}\,8\pi \overset{\text{em}}{T}_{11}\\&=-\frac{\beta B_0^2}{\beta^2+f^2}-\frac{\beta l^4}{\beta^2+f^2}\frac{E^2}{\omega^2}\\&\quad-4\beta\left(\frac{fB_0}{\beta^2+f^2}-\frac{l^4}{\beta^2+f^2}\frac{E}{\omega}\right)^2\end{aligned} \tag{19.22}$$

$$\begin{aligned}8\pi \overset{\text{em}}{T}_{14}&=-8\pi \overset{\text{em}}{T}_{41}\\&=\frac{\omega}{\beta^2+f^2}\left(7l^2\frac{E^2}{\omega^2}-8fB_0\frac{E}{\omega}-B_0^2\right)\\&\quad-\omega\left(\frac{fB_0}{\beta^2+f^2}-\frac{l^4}{\beta^2+f^2}\frac{E}{\omega}\right)^2\end{aligned} \tag{19.23}$$

The rest of the components of $\overset{\text{em}}{T}_{\mu\nu}$ vanish. The electromagnetic Lagrangian in this case is

$$\begin{aligned}\mathscr{L}_{\text{em}}&=\frac{1}{8\pi}\,[2(g^{[\mu\nu]}F_{\mu\nu})^2-H^{\mu\nu}F_{\mu\nu})\\&=\frac{1}{8\pi}\left[\frac{8\omega^4}{(\alpha\gamma-\omega^2)^2}\left(\frac{fB_0}{l^4}-\frac{E}{\omega}\right)^2\right.\\&\quad\left.-\frac{2\omega^2}{\alpha\gamma-\omega^2}\left(\frac{B_0^2}{l^4}-\frac{E^2}{\omega^2}\right)\right]\end{aligned} \tag{19.24}$$

Finally, we have the following equations:

$$A_{11}(\bar{\Gamma})=8\pi \overset{\text{em}}{T}_{11} \tag{19.25a}$$

$$A_{44}(\bar{\Gamma})=8\pi \overset{\text{em}}{T}_{44} \tag{19.25b}$$

$$A_{22}(\bar{\Gamma}) = 8\pi \overset{\text{em}}{T}_{22} \tag{19.25c}$$

$$A_{33}(\bar{\Gamma}) = 8\pi \overset{\text{em}}{T}_{33} \tag{19.25d}$$

$$A_{[23]}(\bar{\Gamma}) - 8\pi \overset{\text{em}}{T}_{23} = C_1 \sin\theta \tag{19.25e}$$

$$A_{(14)}(\bar{\Gamma}) = 0 \tag{19.25f}$$

Using results from Ref. 82 and equation (19.22), one finds the identity (see Appendix)

$$A_{22}(\bar{\Gamma}) - 8\pi \overset{\text{em}}{T}_{22} = \frac{1}{\sin^2\theta}\,[A_{33}(\bar{\Gamma}) - 8\pi \overset{\text{em}}{T}_{33}] \tag{19.26}$$

so that equation (19.25d) is not independent.

In the above

$$\begin{aligned} 8\pi \overset{\text{em}}{T}_{11} = \alpha\{&[4l^2 f B_0 - Q(\beta^2+f^2)]^2 + B_0^2(\beta^2+f^2+4l^4)^2 \\ &- 4(fB_0 + Ql^2)^2(\beta^2+f^2)\} \\ &\times [(\beta^2+f^2)(\beta^2+f^2+4l^4)^2]^{-1} \end{aligned} \tag{19.27}$$

$$\begin{aligned} \frac{8\pi}{\sin\theta}\overset{\text{em}}{T}_{23} &= \frac{8\pi}{\sin\theta}\overset{\text{em}}{T}_{[23]} \\ &= \frac{-7fB_0(\beta^2+f^2+4l^4)^2 - f[4fB_0 - Q(\beta^2+f^2)]^2}{(\beta^2+f^2)(\beta^2+f^2+4l^4)^2} \\ &\quad + \{8B_0 l^4[4B_0 l^2 - Q(\beta^2+f^2)](\beta^2+f^2+4l^2) \\ &\quad + 4f(\beta^2+f^2)(fB_0+Ql^2)^2\} \\ &\quad \times [(\beta^2+f^2)(\beta^2+f^2+4l^4)^2]^{-1} \end{aligned} \tag{19.28}$$

For $\overset{\text{em}}{T}_{14}$ one finds

$$\begin{aligned} 8\pi \overset{\text{em}}{T}_{14} &= 8\pi \overset{\text{em}}{T}_{[14]} \\ &= \frac{\omega}{\beta^2+f^2} \\ &\quad \times \{7l^2[4l^2 f B_0 - Q(\beta^2+f^2)]^2 \\ &\quad - 8B_0 f[4l^2 f B_0 - Q(\beta^2+f^2)] - l^2 B_0(\beta^2+f^2+4l^4)^2\} \\ &\quad \times [l^2(\beta^2+f^2+4l^4)]^{-1} \end{aligned} \tag{19.29}$$

$A_{11}(\bar{\Gamma})$, $A_{44}(\bar{\Gamma})$, $A_{33}(\bar{\Gamma})$, $A_{(14)}(\bar{\Gamma})$, $A_{[14]}(\bar{\Gamma})$, and $A_{[23]}(\bar{\Gamma})$ are given by formulas from Ref. 82. For $\mathscr{L}_{\rm em}$ one easily gets, using (19.24),

$$\mathscr{L}_{\rm em}=\frac{1}{4\pi}\frac{l^4}{\beta^2+f^2}\left[\frac{4}{\beta^2+f^2}\left(\frac{fB_0}{l^2}-\frac{4l^2fB_0-Q(\beta^2+f^2)}{\beta^2+f^2+4l^4}\right)^2\right.$$
$$\left.-\frac{1}{l^4}\left(\frac{[4l^2fB_0-Q(\beta^2+f^2)]^2}{(\beta^2+f^2+4l^4)^2}\right)\right] \tag{19.30}$$

20. STATIC, SPHERICALLY SYMMETRIC SOLUTION

Let us consider a spherical field configuration such that

$$B_0=f=0 \tag{20.1}$$

Later we suppose that

$$\beta=r^2 \tag{20.2}$$

which is simply a coordinate choice. In addition, our quantities do not depend on time (static case). One finds [see equation (19.9)]

$$E=-\omega\frac{Q}{l^2}\frac{r^4}{r^4+4l^4} \tag{20.3}$$

(substituting $\beta=r^2$). Equations (19.29) now read

$$A_{11}(\bar{\Gamma})-\frac{\alpha Q^2(\beta^2-4l^2)}{(\beta^2+4l^4)^2}=0$$
$$A_{44}(\bar{\Gamma})+\frac{\gamma Q^2(\beta^2-4l^4)}{(\beta^2+4l^4)^2}=0$$
$$A_{22}(\bar{\Gamma})-\frac{\beta Q^2(\beta^2-4l^4)}{(\beta^2+4l^4)^2}=0$$
$$A_{(14)}=0$$
$$A_{[23]}-8\pi\overset{\rm em}{T}_{23}=C_1\sin\theta \tag{20.4}$$

and we have

$$8\pi\overset{\rm em}{T}_{[14]}=8\pi\overset{\rm em}{T}_{14}=\omega Q^2\frac{7\beta^2-16l^4}{(\beta^2+4l^4)^2} \tag{20.5}$$

$$\omega=\frac{l^2}{r^2} \tag{20.6}$$

We get

$$E = -\frac{Q}{r^2}\frac{r^4}{r^4+4l^4} \tag{20.7}$$

It is easy to see that the function (20.7) is bounded,

$$|E| \le E_{\max} = |E(\sqrt{2}\,l)| = \frac{|Q|}{4l^2} \tag{20.7a}$$

Taking the linear combination

$$\frac{1}{\alpha}A_{11} + \frac{1}{\gamma}A_{44} = 0$$

one finds

$$\frac{d}{dr}[\log(\alpha\gamma)] = -\frac{4}{r}\frac{l^4}{l^4+r^4} \tag{20.8}$$

which gives

$$\alpha\gamma = B\left(1+\frac{l^4}{r^4}\right) \tag{20.8a}$$

where B is a constant of integration. Taking $B=1$ and substituting (20.8a) into the third equation of (20.4) yields

$$\frac{d}{dr}(r\alpha^{-1}) = 1 - Q^2\frac{r^2}{r^4+4l^4} \tag{20.8b}$$

Thus, we have

$$\frac{1}{\alpha} = 1 + \frac{C}{r} + \frac{Q^2}{r}K(r,l) \tag{20.9}$$

where

$$K(r,l) = -\int \frac{r^4}{r^4+4l^4}\,dr \tag{20.10}$$

and C is a constant of integration. Moreover,

$$\gamma = \left(1 + \frac{C}{r} + \frac{Q^2}{r}K(r,l)\right)\left(1+\frac{l^4}{r^4}\right) \tag{20.11}$$

Performing the integration in (20.10), one gets

$$\frac{1}{\alpha}=1+\frac{C}{r}+\frac{Q^2}{br}g\left(\frac{r}{b}\right) \tag{20.12}$$

where $b^4=4l^4$ and

$$g(x)=\frac{1}{4\sqrt{2}}\log\left(\frac{x^2+\sqrt{2}\,x+1}{x^2-\sqrt{2}\,x+1}\right)$$

$$-\frac{1}{2\sqrt{2}}[\mathrm{arctg}(\sqrt{2}\,x+1)+\mathrm{arctg}(\sqrt{2}\,x-1)] \tag{20.13}$$

The function $g(x)$ is plotted on Figure 2. Let us examine the properties of the function

$$\frac{1}{r}g\left(\frac{r}{b}\right)=\tilde{g}\left(\frac{r}{b}\right)$$

It can be shown that

$$\lim_{r=0}\tilde{g}\left(\frac{r}{b}\right)=0 \tag{20.14}$$

Thus, for small r we get

$$\alpha^{-1}\simeq 1+\frac{C}{r} \tag{20.15}$$

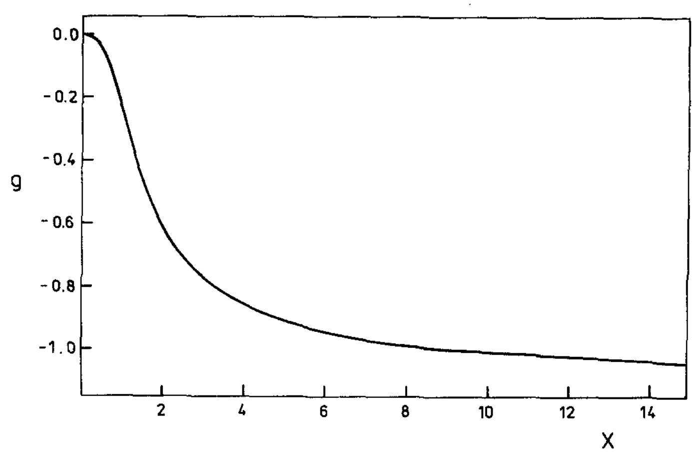

Fig. 2. The function $g=g(x)$ versus x [equation (20.13)].

We can avoid a singularity in α at $r=0$ by choosing

$$C=0 \tag{20.16}$$

so that

$$\lim_{r=0} \alpha^{-1}=1 \tag{20.17}$$

Let us examine the asymptotic properties of α and γ. We get

$$\alpha^{-1} \xrightarrow[r\to\infty]{} 1-\frac{2m_N}{r}+\frac{Q^2}{r^2} \tag{20.18}$$

For large r, α clearly behaves like the analogous function in the Reissner–Nördström solution, with Q as the electric charge and with

$$c^2 m_N=\frac{\pi Q^2}{2\sqrt{2}\,b} \tag{20.19}$$

playing the role of the Newtonian mass. To summarize, we have

$$\alpha^{-1}=1+\frac{Q^2}{br}\,g\left(\frac{r}{b}\right) \tag{20.20}$$

where

$$\lim_{r\to\infty} g\left(\frac{r}{b}\right)=0 \tag{20.21}$$

and

$$\lim_{r\to\infty} \alpha^{-1}=1 \tag{20.22}$$

In the neighborhood of $r=0$ one gets for our metric

$$\mathbf{g}_{\mu\nu}=\begin{bmatrix} -1 & 0 & 0 & l^2/r^2 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin^2\theta & 0 \\ -l/r & 0 & 0 & 1+l^4/r^4 \end{bmatrix} \tag{20.23}$$

(for $r\to 0$). The determinant of the symmetric part of the metric is

$$(-\tilde{\mathbf{g}})^{1/2}=(r^4+l^4)^{1/2}\sin\theta \tag{20.24}$$

The full determinant is

$$(-\mathbf{g})^{1/2}=r^2\sin\theta \tag{20.25}$$

Thus, there is a singularity at $r=0$. It is worth noting, however, that there is no singularity in α and only one singularity in γ, due to the $(1+l^4/r^4)$ factor. Note that ω, the skew-symmetric part of $\mathbf{g}_{\mu\nu}$, is also singular at $r=0$.

Let us examine the properties of the electric field:

$$E=-\frac{Q}{r^2}\frac{r^4}{r^4+4l^4} \tag{20.26}$$

One easily sees that

$$E(0)=0 \tag{20.27}$$

and

$$E \xrightarrow[r\to\infty]{} -\frac{Q}{r^2} \tag{20.28}$$

Thus, there is no singularity at $r=0$. This is similar to the situation in Born–Infeld electrodynamics.[87] Let us calculate the charge for the electric field. It is known that

$$4\pi\sqrt{-\mathbf{g}}\,\rho=\underset{\sim}{H}^{4i}{}_{,i}=\operatorname{div}\mathbf{D} \tag{20.29}$$

where ρ is the charge density distribution and $\mathbf{D}$ is the electric induction vector. One gets

$$\underset{\sim}{H}^{41}=\sqrt{-\mathbf{g}}\,\frac{E}{\alpha\gamma-\omega^2}=\sqrt{-\mathbf{g}}\,E \tag{20.30}$$

and

$$\sqrt{-\mathbf{g}}\,\rho=-\frac{1}{\pi}\frac{Q}{r}\frac{4l^4r^4}{(r^4+4l^4)^2}\sin\theta \tag{20.31}$$

The total charge is

$$Q_{\text{tot}}=\int_{R^3}\sqrt{-\mathbf{g}}\,\rho\,d^3x=-16Ql^4\int_0^\infty\frac{1}{r}\frac{r^4}{(r^4+4l^4)^2}\,dr=-Q \tag{20.32}$$

Thus, we find the following interesting feature: the total electric charge defined above is the same as the charge obtained from the asymptotic properties of the electric field E and the metric (functions α and γ). Let us pass to the calculation of the energy of the electromagnetic field. We have

$$\tfrac{1}{2}(\mathbf{g}^{4\mu}T_{4\mu}+\mathbf{g}^{\mu4}T_{\mu4})=\overset{\text{em}}{T}{}^4{}_4=\frac{1}{8\pi}Q^2\frac{1}{r^4+4l^4} \tag{20.33}$$

The total energy is given by

$$\frac{1}{c^2} E_{\text{tot}} = 4\pi \int_0^\infty r^2 \overset{\text{em}}{T^4}_4 \, dr = \frac{\pi}{2\sqrt{2}} \frac{Q^2}{b} = m_N \tag{20.34}$$

where $b^4 = 4l^4$.

This energy can be treated as the energy of the electric field of the charge Q distributed over a sphere of radius r_0. That is,

$$c^2 m_N^2 = c^2 m_{\text{em}} = \frac{Q^2}{r_0} \tag{20.35}$$

so that

$$r_0 = \frac{4}{\pi c^2} l \tag{20.36}$$

Let us suppose that the mass m_N is the mass of an electron,

$$m_N = m_e \qquad \text{and} \qquad Q = e \tag{20.37}$$

We get, where e is the elementary charge,

$$m_e c^2 = \frac{\pi}{2\sqrt{2}} \frac{Q^2}{b} \tag{20.38}$$

Thus, we get

$$l = \frac{\pi}{4} \frac{e^2}{m_e c^2} = \frac{\pi r_{\text{cl}}}{4} \tag{20.39}$$

where the classical radius of the electron is defined as

$$r_{\text{cl}} = \frac{e_2}{m_e c^2} \simeq 2.81 \times 10^{-13} \, [\text{cm}] \tag{20.40}$$

Let us introduce the following dimensionless variables:

$$q \equiv \frac{Q}{b} = \frac{Q}{l\sqrt{2}} \tag{20.41}$$

$$R \equiv \frac{r}{b} \tag{20.42}$$

Using equations (20.41) and (20.42), we have

$$\alpha^{-1} = 1 + \frac{g^2}{R} g(R) = 1 - q^2 P(R) \tag{20.43}$$

$$E = -\frac{q^2}{R^2} \frac{R^4}{R^4+1} = q^2 \tilde{E} \tag{20.44}$$

$$e = 4\pi \overset{\text{em}}{T}{}^4{}_4 r^2 = \frac{q^2 R^2}{2} \frac{1}{R^4+1} = q^2 \tilde{e} \tag{20.45}$$

$$\rho_R = \frac{4\pi \rho r^2}{b^4} = -\frac{2q}{R} \frac{R^4}{R^4+1} = q\tilde{\rho}_R \tag{20.46}$$

where q is the normalized charge, R is the normalized radial coordinate, and $\tilde{E}$, $\tilde{e}$, and $\tilde{\rho}_R$ are normalized electric field, radial energy distribution, and radial charge distribution, respectively. These functions are plotted in Figures 3–5. Recall that the radial charge distribution of our solution is similar to the radial charge distribution for Abraham's model of the electron,[(89)] where the charge is distributed on a sphere of radius r_{cl}. In our case gravitational forces play the role of Abraham's elastic forces. The function

$$P(R) = -\frac{1}{R} g(R) \tag{20.47}$$

is plotted in Figure 6. It expresses the properties of the generalized Newtonian potential for our solution.

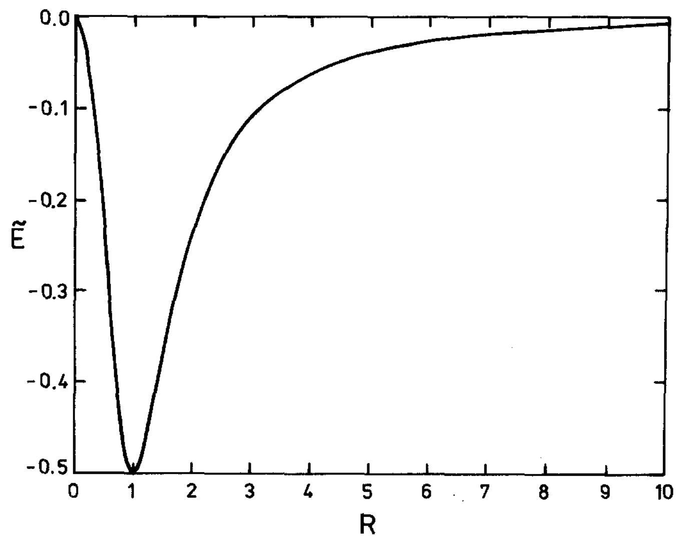

Fig. 3. The function $\tilde{E} = \tilde{E}(R)$ versus R (normalized electric field)

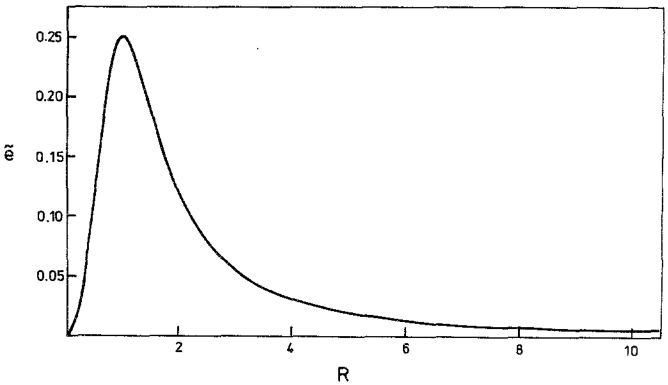

Fig. 4. The function $\tilde{e}=\tilde{e}(R)$ versus R (normalized radial energy distribution) [equation (2.45)].

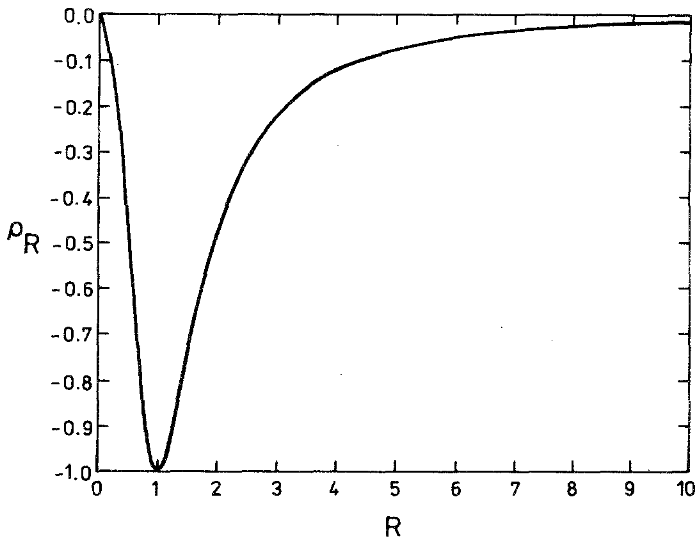

Fig. 5. The function $\tilde{\rho}_R=\tilde{\rho}_r(R)$ versus R (normalized radial charge distribution) [equation (20.46)].

An interesting question which we can pose here concerns the existence of event horizons. This problem reduces to finding real roots for the function $\alpha^{-1}=f(R, q)$. This depends of course on the value of the parameter q. Let us consider the function

$$f(R, q)=1+q^2\frac{1}{R}g(R) \tag{20.48}$$

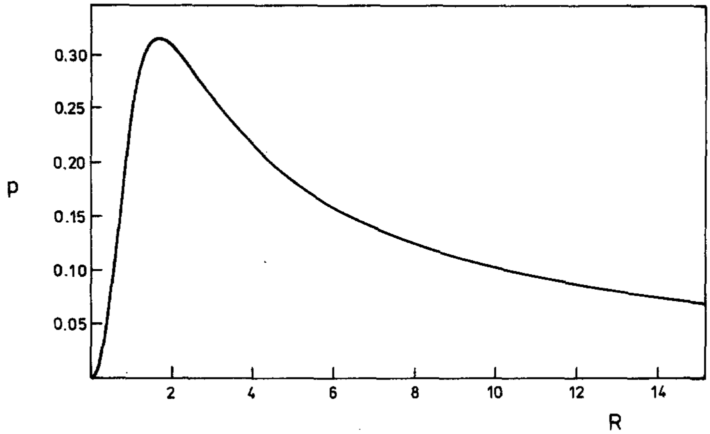

Fig. 6. The function $P = P(R)$ versus R (generalized Nördström function) [equation (20.47)].

We have

$$f(0, q) = 1 \tag{20.49a}$$

and

$$\lim_{R \to \infty} f(R, q) = 1 \tag{20.49b}$$

The function $g(R)$ is monotonic and negative in the interval $\langle 0, \infty)$. Let us take $R_1 \in (0, \infty)$; we have

$$\frac{g(R_1)}{R_1} < 0 \tag{20.50}$$

Let us suppose that

$$q > \left[-\frac{R_1}{g(R_1)} \right]^{1/2} \tag{20.51}$$

It is easy to check that if (20.51) is satisfied, then

$$f(q, R_1) < 0 \tag{20.52}$$

Thus, the function changes sign in the interval $(0, R_1\rangle$. This means that there exists a value $R_{H_1} \in (0, R_1)$ such that

$$f(q, R_{H_1}) = 0 \tag{20.53}$$

The function $f(q, R)$ also changes sign in the interval $\langle R_1, +\infty)$. Thus, there exists a value $R_{H_2} \in (R_2, +\infty)$ such that

$$f(q, R_{H_2}) = 0, \qquad R_{H_1} < R_1 < R_{H_2}$$

[if condition (20.51) is satisfied]. Moreover, the function $f(q, R)$ has one minimum regardless of the value of q. Thus, it can cross a horizontal axis two times at most. Hence there are two event horizons for sufficiently large q in general.

Let us examine the situation with only one event horizon. The conditions necessary for the existence of a single horizon are as follows:

$$f(q, R) = 0 \tag{20.54a}$$

$$\frac{df}{dR}(q, R) = 0 \tag{20.54b}$$

One easily gets

$$g(R_0) = -\frac{R_0^3}{R_0^4 + 1} \tag{20.55}$$

$$q_0 = \frac{(R_0^4 + 1)^{1/2}}{R_0} \tag{20.56}$$

From equation (20.54) one gets

$$R_0 = 1.6787\ldots \tag{20.57a}$$

$$q_0 = 1.78126\ldots \tag{20.57b}$$

Thus,

$$r_H = R_0 b = \sqrt{2}\, R_0 l = 2.37 l$$

In this case the charge Q and the mass m_N are

$$\begin{aligned} Q_0 &= q_0 b = 2.53 l \\ m_N^0 &= \frac{\pi (R_0^4 + 1)}{4 R_0^2}\, l = 2.48 l \end{aligned} \tag{20.58}$$

For $l = 10^{-22}$ cm the total charge is

$$Q_0 = 2.53 \frac{l c^2}{\sqrt{G_N}} \simeq 10^{15}\ \text{esu} \simeq 10^{14}\ \text{elementary charges}$$

$$m_N^0 = 2.48 \frac{c^2 l}{G_N} \simeq 10^7\ \text{g}$$

It is easy to see that if $q > q_0$, we have two horizons. This also implies that

$$m_N > m_N^0 \tag{20.59}$$

In other words, the Newtonian mass is large enough to form event horizons. If $q = q_0$, we have only one horizon and if $q < q_0$, we have no horizons. This situation is described in Figure 7, where we plot the function $\alpha^{-1} = f(q, R)$ for various values of the parameter q.

For example, for an electron one has

$$q_{\text{electron}} = \frac{e\sqrt{G_N}}{\sqrt{2}\, lc^2} \simeq 10^{-37} \ll q_0 \tag{20.60}$$

Thus, there are no event horizons. It is worth noting that if there exists only one event horizon, the solution is unstable due to pair creation and Hawking radiation. Such "black holes" are "very hot"[89] and decay very quickly. In the case of two event horizons the solution is unstable because of pair creation. If the Newtonian mass is sufficiently big, this solution could be more stable because the Hawking effect is not important for very massive black holes.[89] The situation without any event horizons is very interesting from the physical point of view, because it corresponds to the parameter q for electrons (in general, for any elementary particle). Thus, we have in this case a singularity without a horizon. The structure of this singularity is different from the Reissner–Nördström singularity in general relativity and

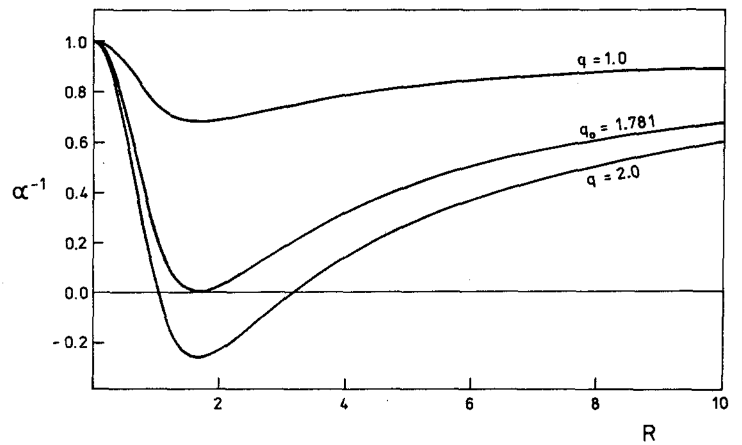

Fig. 7. The function $\alpha^{-1} = f(q, R)$ versus R for various values of parameters q. Here q_0 denotes the critical value for which we have only one event horizon for the value $R = R_H$. For the value $R = R_H$ the function $f(q, R)$ has a minimum regardless of the value of q. If $q > q_0$, we have two event horizons [two real roots of $f(q, R)$ R_{H1}, $R_H 2$]. If $q < q_0$, there are no event horizons [no real roots for $f(q, R)$].

the Reissner–Nördström-like or Schwarzschildlike singularity in the nonsymmetric theory of gravitation [see Refs. 81 and 63 and equation (20.23)].

To summarize, we have found the following exact solution (in the form suggested in Section 6 of Ref. 18):

$$\mathbf{g}_{\mu\nu}=\begin{bmatrix} -\alpha & 0 & 0 & l^2/r^2 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2\sin^2\theta & 0 \\ -l^2/r^2 & 0 & 0 & \gamma \end{bmatrix} \tag{20.61}$$

$$\alpha=\left(1+\frac{Q^2}{rb}g\left(\frac{r}{b}\right)\right)^{-1} \tag{20.62}$$

$$\gamma=\left(1+\frac{l^4}{r^4}\right)\left(1+\frac{Q^2}{rb}g\left(\frac{r}{b}\right)\right) \tag{20.63}$$

$$b^4=4l^4 \tag{20.64}$$

$$E=-\frac{Q}{r^2}\left(\frac{r^2}{r^4+4l^4}\right) \tag{20.65}$$

The function g is plotted on Figure 2 [see equation (20.13)]. The solution has one horizon if

$$Q=Q_0=2.53\,\frac{lc^2}{\sqrt{G_N}} \tag{20.66}$$

If $Q<Q_0$, there are no horizons. If $Q>Q_0$, we have two horizons (as for the Reissner–Nördström solution to the Einstein–Maxwell equations). In other words, the horizons exist if the mass is sufficiently big [see equation (20.59)]. Finally, let us calculate the ratio Q/m_N for our solution. We get, using equation (20.34),

$$\frac{Q}{m_N}=\frac{4\sqrt{2}}{\pi q} \tag{20.67}$$

Finally, let us check the generalized Bianchi identity for our solution. We have

$$(\underset{\sim}{\mathbf{g}}^{\nu\beta}T_{\nu\gamma}+\underset{\sim}{\mathbf{g}}^{\beta\nu}T_{\gamma\nu})_{,\alpha}+\sqrt{-\mathbf{g}}\,\mathbf{g}^{\beta\delta}{}_{,\gamma}\,T_{\beta\delta}=0 \tag{20.68}$$

In our case (spherical symmetry and static) one derives from equation (20.68) the simpler expression

$$\frac{d}{dr}(\underset{\sim}{g}^{11}T_{11})-\tfrac{1}{2}\underset{\sim}{g}^{(\alpha\beta)}\frac{d}{dr}T_{(\alpha\beta)}=0 \tag{20.69}$$

and only the symmetric part of the energy-momentum tensor enters. Substituting $T_{(\mu\nu)}$ from equation (20.4), one easily gets the desired identity. Equation (20.68) can be derived from the Bianchi identity for $\bar{R}_{\alpha\beta}(\bar{\Gamma})$ or $\bar{R}_{\alpha\beta}(\bar{W})$ in our theory using the field equations and it expresses the energy-momentum conservation laws. Thus, our solution satisfies the energy-momentum conservation laws.

21. TEST PARTICLE MOTION IN THE EXACT SOLUTION IN NKKT

In this section we consider equations of motion for test particles in space-time described by our solution.

Let us calculate the connection $\bar{\Gamma}^{\alpha}_{\beta\gamma}$ and the Christoffel symbols for our solution. We get (using results from Ref. 82)

$$\begin{aligned}
&\bar{\Gamma}^{1}_{[14]}=\frac{2l^2}{\alpha r^3}\\
&\bar{\Gamma}^{2}_{33}=-\tfrac{1}{2}\sin 2\theta\\
&\bar{\Gamma}^{3}_{23}=\bar{\Gamma}^{3}_{32}=\operatorname{ctg}\theta\\
&\bar{\Gamma}^{1}_{22}=\frac{1}{\sin^2\theta}\Gamma^{1}_{33}=-\frac{r}{\alpha}\\
&\bar{\Gamma}^{2}_{(12)}=\bar{\Gamma}^{3}_{(13)}=\frac{1}{r}\\
&\bar{\Gamma}^{2}_{[24]}=\bar{\Gamma}^{3}_{[34]}=-\frac{l^2}{\alpha r^3}\\
&\bar{\Gamma}^{1}_{11}=\frac{\alpha'}{2\alpha}\\
&\Gamma^{1}_{44}=\frac{4l^4}{r^5\alpha^2}+\frac{\gamma'}{2\alpha}=\frac{7l^2}{8\alpha^2 r^5}-\left(1+\frac{l^4}{r^4}\right)\frac{\alpha'}{2\alpha^3}\\
&\bar{\Gamma}^{4}_{(14)}=\frac{2l^4}{r^5\alpha}+\frac{\gamma'}{2\gamma}=\frac{3l^4}{2r^4}\left(1+\frac{l^4}{r^4}\right)^{-1}-\frac{\alpha'}{2\alpha}
\end{aligned} \tag{21.1}$$

The remaining $\bar{\Gamma}$'s are zero. Let us consider the symmetric part of our solution, i.e.,

$$\mathbf{g}_{(\mu\nu)}=\begin{bmatrix}-\alpha & 0 & 0 & 0\\ 0 & -r^2 & 0 & 0\\ 0 & 0 & -r^2\sin^2\theta & 0\\ 0 & 0 & 0 & \gamma\end{bmatrix} \tag{21.2}$$

where α and γ are given by formulas (20.62) and (20.63). One easily finds the determinant

$$\tilde{\mathbf{g}}=\det[\mathbf{g}_{(\mu\nu)}]=-\left(1+\frac{l^4}{r^4}\right)r^4\sin^2\theta \tag{21.3}$$

The determinant is not singular at $r=0$. The inverse tensor for $\mathbf{g}_{(\mu\nu)}$

$$\tilde{\mathbf{g}}^{(\mu\alpha)}\mathbf{g}_{(\alpha\nu)}=\delta^{\mu}_{\nu} \tag{21.4}$$

is the following:

$$\tilde{\mathbf{g}}^{(\mu\nu)}=\begin{bmatrix}-1/\alpha & 0 & 0 & 0\\ 0 & -1/r^2 & 0 & 0\\ 0 & 0 & -1/(r^2\sin^2\theta) & 0\\ 0 & 0 & 0 & \gamma\end{bmatrix} \tag{21.5}$$

Let us calculate the Christoffel symbols for $\mathbf{g}_{(\mu\nu)}$;

$$\left\{\begin{matrix}\alpha\\ \beta\end{matrix}\right\}=\tfrac{1}{2}\tilde{\mathbf{g}}^{(\alpha\mu)}(\mathbf{g}_{(\beta\mu),\gamma}+\mathbf{g}_{(\gamma\mu),\beta}-\mathbf{g}_{(\beta\gamma),\mu}) \tag{21.6}$$

We easily find

$$\left\{\begin{matrix}1\\ 11\end{matrix}\right\}=\frac{\alpha'}{2\alpha}$$

$$\left\{\begin{matrix}1\\ 22\end{matrix}\right\}=\frac{r}{\alpha}$$

$$\left\{\begin{matrix}1\\ 33\end{matrix}\right\}=\frac{r}{\alpha}\sin^2\theta$$

$$\left\{\begin{matrix}2\\ 33\end{matrix}\right\}=-\tfrac{1}{2}\sin 2\theta$$

$$\left\{ {3 \atop 32} \right\} = \operatorname{ctg}\theta$$

$$\left\{ {1 \atop 44} \right\} = \frac{\gamma'}{2\alpha} = \frac{-l^4}{2\alpha^2 r^5} - \left(1 + \frac{l^4}{r^4}\right)\frac{\alpha'}{2\alpha^3}$$

$$\left\{ {2 \atop 21} \right\} = \frac{1}{r} = \left\{ {3 \atop 31} \right\}$$

$$\left\{ {4 \atop 41} \right\} = -\frac{\gamma'}{2\gamma} = \frac{\alpha'}{2\alpha} + \frac{l^4}{r^4}\left(1 + \frac{l^4}{r^4}\right)^{-1} \tag{21.7}$$

The remaining Christoffel symbols are zero. Let us write the equation of motion for an uncharged test particle for our solution, i.e., the equation for geodesics,

$$\frac{d^2 x^\alpha}{d\tau^2} + \bar{\Gamma}^\alpha{}_{(\beta\gamma)} \frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau} = 0 \tag{21.8}$$

We easily find from (21.1)

$$\frac{d^2 r}{d\tau^2} + \frac{\alpha'}{2\alpha}\left(\frac{dr}{d\tau}\right)^2 + \left[\frac{7l^4}{8\alpha^2 r^5} - \left(1 + \frac{l^4}{r^4}\right)\frac{\alpha'}{2\alpha^3}\right]\left(\frac{dt}{d\tau}\right)^2$$
$$-\frac{r}{\alpha}\left[\left(\frac{d\theta}{d\tau}\right)^2 + \sin^2\theta\left(\frac{d}{d\tau}\right)^2\right] = 0$$

$$\frac{d^2\theta}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau} - \frac{\sin 2\theta}{2}\left(\frac{d\phi}{d\tau}\right)^2 = 0 \tag{21.9}$$

$$\frac{d^2\phi}{dt^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\phi}{d\tau} + 2\operatorname{ctg}\theta\frac{d\phi}{d\tau}\frac{d\theta}{d\tau} = 0$$

$$\frac{d^2 t}{d\tau^2} + \left[\frac{3l^4}{2r(l^4 + r^4)} - \frac{\alpha'}{2\alpha}\right]\frac{dr}{d\tau}\frac{dt}{d\tau} = 0$$

In the nonsymmetric theory of gravitation uncharged particles move along geodesics in Riemannian geometry formed from $g_{(\mu\nu)}$,[34] i.e., in Christoffel symbols,

$$\frac{d^2 x^\alpha}{d\tau^2} + \left\{ {\alpha \atop \beta} \right\}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0 \tag{21.10}$$

One easily finds from (21.7)

$$
\begin{gathered}
\frac{d^2r}{d\tau^2}+\frac{\alpha'}{2\alpha}\left(\frac{dr}{d\tau}\right)^2-\left[\frac{l^4}{2\alpha^2r^5}+\left(1+\frac{l^4}{r^4}\right)\frac{\alpha'}{2\alpha^3}\right]\left(\frac{dt}{d\tau}\right)^2 \\
+\frac{r}{\alpha}\left[\left(\frac{d\theta}{d\tau}\right)^2+\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right]=0 \\
\frac{d^2\theta}{d\tau^2}-\frac{\sin 2\theta}{2}\left(\frac{d\phi}{d\tau}\right)^2+\frac{2}{r}\frac{d\theta}{d\tau}\frac{dr}{d\tau}=0 \\
\frac{d^2\phi}{d\tau^2}+\frac{2}{r}\frac{d\phi}{d\tau}\frac{dr}{d\tau}+2\operatorname{ctg}\theta\frac{d\phi}{d\tau}\frac{d\theta}{d\tau}=0 \\
\frac{d^2t}{d\tau^2}+\left[\frac{\alpha'}{2\alpha}+\frac{l^4}{r(l^4+r^4)}\right]\frac{dt}{d\tau}\frac{d\theta}{d\tau}=0
\end{gathered}
\tag{21.11}
$$

Let us find the equations of motion for a charged test particle. In the nonsymmetric Kaluza–Klein theory (NKKT) (see Section 9)

$$
\frac{d^2x^\alpha}{d\tau^2}+\bar{\Gamma}^\alpha_{(\beta\gamma)}\frac{dx^\beta}{d\tau}\frac{dx^\gamma}{d\tau}+\left(\frac{q}{m_0}\right)[\mathrm{g}^{\alpha\gamma}F_{\gamma\beta}-\mathrm{g}^{[\alpha\gamma]}H_{\gamma\beta}]\frac{dx^\beta}{d\tau}=0 \tag{21.12}
$$

where q is the charge and m_0 the rest mass of a test particle. Using (20.9) and (20.7), one gets

$$
\begin{gathered}
\frac{d^2r}{d\tau^2}+\frac{\alpha'}{2\alpha}\left(\frac{dr}{d\tau}\right)^2+\left[\frac{7l^4}{2\alpha^2r^5}-\left(1+\frac{l^4}{r^4}\right)\frac{\alpha'}{2\alpha^3}\right]\left(\frac{dt}{d\tau}\right)^2 \\
-\frac{r}{\alpha}\left[\left(\frac{d\theta}{d\tau}\right)^2+\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right] \\
-\frac{q}{m_0}\frac{Q}{\alpha r^2}\frac{r^4+l^4}{r^4+4l^4}\frac{dt}{d\tau}=0 \\
\frac{d^2\theta}{d\tau^2}+\frac{2}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau}-\frac{\sin 2\theta}{2}\frac{d\phi}{d\tau}=0 \\
\frac{d^2t}{d\tau^2}+\frac{3l^4}{2r(l^4+r^4)}\frac{dr}{d\tau}\frac{dt}{d\tau}+\frac{q}{m_0}\frac{r^2\alpha Q}{r^4+4l^4}\frac{dr}{d\tau}=0
\end{gathered}
\tag{21.13}
$$

In Ref. 25 and in Section 12 a different possibility is considered for the equations of motion for a charged test particle,

$$
\frac{d^2x^\alpha}{d\tau^2}+\left\{\begin{matrix}\alpha\\ \beta\gamma\end{matrix}\right\}\frac{dx^\beta}{d\tau}\frac{dx^\alpha}{d\tau}+\frac{q}{m_0}[\mathrm{g}^{\alpha\gamma}F_{\gamma\beta}-\mathrm{g}^{[\alpha\gamma]}H_{\gamma\beta}]\frac{dx^\beta}{d\tau}=0 \tag{21.14}
$$

Using (21.9) and (21.11), one finds the equations

$$\frac{d^2r}{d\tau^2}+\frac{\alpha'}{2\alpha}\left(\frac{dr}{d\tau}\right)^2-\left[\frac{l^4}{2\alpha^2r^5}+\left(1+\frac{l^4}{r^4}\right)\frac{\alpha'}{2\alpha^3}\right]\left(\frac{dt}{d\tau}\right)^2+\frac{r}{\alpha}\left[\left(\frac{d\theta}{d\tau}\right)^2+\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right]$$
$$-\frac{q}{m_0}\frac{Q}{\alpha r^2}\frac{r^4+l^4}{r^4+8l^4}\frac{dt}{d\tau}=0$$

$$\frac{d^2\theta}{d\tau^2}-\frac{\sin 2\theta}{2}\left(\frac{d\phi}{d\tau}\right)^2+\frac{2}{r}\frac{d\theta}{d\tau}\frac{dr}{d\tau}=0$$

$$\frac{d^2\phi}{d\tau^2}+\frac{2}{r}\frac{d\phi}{d\tau}\frac{dr}{d\tau}+2\operatorname{ctg}\theta\frac{d\phi}{d\tau}\frac{d\phi}{d\tau}=0 \tag{21.15}$$

$$\frac{d^2t}{d\tau^2}+\left[\frac{\alpha'}{2\alpha}+\frac{l^4}{r(l^4+r^4)}\right]\frac{dt}{d\tau}\frac{dr}{d\tau}+\frac{q}{m_0}\frac{r^2\alpha Q}{r^4+8l^4}\frac{dr}{d\tau}=0$$

Notice that the equations for θ and ϕ are the same in (21.9), (21.11), (21.13), and (21.15) regardless of the connections and whether the particle is charged or not. For α' we have

$$\alpha'=\frac{\alpha}{r}+\alpha^2\left(\frac{Q^2r^2}{r^4+4l^4}-\frac{1}{r}\right) \tag{21.16}$$

where α is given by (20.13). According to the general properties of the geodetic equations in Einstein's unified field theory, the nonsymmetric theory of gravitation, and the nonsymmetric Kaluza–Klein theory, equations (20.9), (21.11), (21.3), and (21.15) have the following first integral of motion (see Ref. 25 and Section 9):

$$\gamma\left(\frac{dt}{d\tau}\right)^2-\alpha\left(\frac{dr}{d\tau}\right)^2-r^2\left[\left(\frac{d\theta}{d\tau}\right)^2+\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right]=\text{const} \tag{21.17}$$

We can choose const = 1, i.e., we consider timelike world-lines and

$$\gamma\left(\frac{dt}{d\tau}\right)^2-\alpha\left(\frac{dr}{d\tau}\right)^2-r^2\left[\left(\frac{d\theta}{d\tau}\right)^2+\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right]=1 \tag{21.18}$$

Let us consider equations for θ and ϕ:

$$\begin{aligned}&\frac{d^2\theta}{d\tau^2}-\frac{\sin 2\theta}{2}\left(\frac{d\phi}{d\tau}\right)^2+\frac{2}{r}\frac{d\theta}{d\tau}\frac{dr}{d\tau}=0\\&\frac{d^2\phi}{d\tau^2}+2\operatorname{ctg}\theta\frac{d\phi}{d\tau}\frac{d\theta}{d\tau}+\frac{2}{r}\frac{d\phi}{d\tau}\frac{dr}{d\tau}=0\end{aligned} \tag{21.19}$$

We easily find the first integral of motion of (21.10),

$$r^2\left[\left(\frac{d\theta}{d\tau}\right)^2+\sin^2\theta\left(\frac{d\phi}{d\tau}\right)^2\right]=\frac{2E_0}{r^2} \tag{21.20}$$

where

$$E_0=\text{const} \tag{21.21}$$

Comparing (21.18) and (21.20), we get

$$\left(\frac{dt}{d\tau}\right)^2-\alpha\left(\frac{dr}{d\tau}\right)^2=1-\frac{2E_0}{r^2} \tag{21.22}$$

Let us consider the second equation of (21.19). We easily find the following first integral of motion:

$$\frac{d\phi}{d\tau}=\frac{L}{r^2\sin^2\theta} \tag{21.23}$$

where $L=\text{const}$. Comparing (21.20) and (21.23), we get

$$\left(\frac{d\theta}{d\tau}\right)^2=\frac{1}{r^4}\left(2E_0-\frac{L^2}{\sin^2\theta}\right) \tag{21.24}$$

The first integrals (21.20) and (21.22) lead to the following simplifications of equations (21.9), (21.11), (21.13), and (21.15):

$$\frac{d^2r}{d\tau^2}-\frac{7l^4}{8r(l^4+r^4)}\left(\frac{dr}{d\tau}\right)^2-\left[\frac{7l^4}{8r\alpha(l^4+r^4)}+\frac{\alpha'}{2\alpha^2}\right]\left(1-\frac{2E_0}{r^2}\right)-\frac{2E_0}{\alpha r^3}=0 \tag{21.9a}$$

$$\frac{d^2t}{d\tau^2}+\left[\frac{3l^4}{2r(l^4+r^4)}-\frac{\alpha'}{2\alpha}\right]\frac{dr}{d\tau}\frac{dt}{d\tau}=0$$

$$\frac{d^2r}{d\tau^2}-\frac{l^4}{2r(l^4+r^4)}\left(\frac{dr}{d\tau}\right)^2-\left[\frac{l^4}{2r\alpha(l^4+r^4)}+\frac{\alpha'}{2\alpha^2}\right]\left(1-\frac{2E_0}{r^2}\right)+\frac{2E_0}{\alpha r^3}=0 \tag{21.11a}$$

$$\frac{d^2t}{d\tau^2}+\left[\frac{\alpha'}{2\alpha}+\frac{l^4}{r(l^4+r^4)}\right]\frac{dt}{d\tau}\frac{dr}{d\tau}=0$$

$$\frac{d^2r}{d\tau^2}+\frac{7l^4}{8r(l^4+r^4)}\left(\frac{dr}{d\tau}\right)^2+\left[\frac{7l^4}{8\alpha r(l^4+r^4)}-\frac{\alpha'}{2\alpha}\right]\left(1-\frac{2E_0}{r^2}\right)-\frac{2E_0}{\alpha r^3}$$

$$-\frac{q}{m_0}\frac{Q}{\alpha r^2}\frac{r^4+l^4}{r^4+4l^4}\frac{dt}{d\tau}=0$$

$$\frac{d^2t}{d\tau^2}+\left(\frac{3l^4}{2r(l^4+r^4)}-\frac{\alpha'}{2\alpha}\right)\frac{dr}{d\tau}\frac{dt}{d\tau}+\frac{q}{m_0}\frac{r^2\alpha Q}{r^4+4l^4}\frac{dr}{d\tau}=0 \qquad (21.13a)$$

$$\frac{d^2r}{d\tau^2}-\frac{l^4}{2r(l^4+r^4)}\left(\frac{dr}{d\tau}\right)^2-\left[\frac{l^4}{2r\alpha(l^4+r^4)}+\frac{\alpha'}{2\alpha^2}\right]\left(1-\frac{2E_0}{r^2}\right)+\frac{2E_0}{\alpha r^3}$$

$$-\frac{q}{m_0}\frac{Q}{\alpha r^2}\frac{r^4+l^4}{r^4+4l^4}\frac{dt}{d\tau}=0$$

$$\frac{d^2t}{d\tau^2}+\left[\frac{\alpha'}{2\alpha}+\frac{l^4}{r(l^4+r^4)}\right]\frac{dt}{d\tau}\frac{dr}{d\tau}+\frac{q}{m_0}\frac{r^2\alpha Q}{r^4+4l^4}\frac{dr}{d\tau}=0 \qquad (21.15a)$$

For angular coordinates we have for equations (20.9a), (21.11a), (21.13), and (21.15a) the same equations (21.19) and the same first integrals of motion (21.20), (21.22), and (21.23). In this section we consider timelike world-lines of a charged test particle ($m_0 \neq 0$). Moreover, taking $m_0=q=0$ and const $=0$ in equation (21.17), one gets extremal and nonextremal null trajectories on E.

It would be interesting to examine the geodetic completeness of our solution for Einstein and Riemann connections. This will be done elsewhere.

22. SUMMARY OF THE PROPERTIES OF THE SOLUTION AND PROSPECTS

We have found an exact static, spherically symmetric solution for the nonsymmetric Kaluza–Klein theory (NKKT).[18,25] Our solution has the following properties: The metric (symmetric part of $\mathbf{g}_{\alpha\beta}$) behaves asymptotically like the Reissner–Nördström solution of general relativity (apart from a factor of $1+l^4/r^4$), which is typical of the nonsymmetric gravitational theory.[63,81] The most remarkable feature of this metric is that the function α is not singular at $r=0$ and goes to 1 as $r \to 0$. We have calculated the total energy of the solution, which is its Newtonian mass. This quantity is constructed from q and l, the charge and fermion number parameters, respectively. The electric field of our solution behaves asymptotically like the Coulomb field generated by a charge Q. Moreover, this field vanishes at $r=0$ and is nonsingular for all r. We get a maximal value of this field similar to that in Born–Infeld electrodynamics.[87,90] We calculated the charge distribution for such a field and showed that it is nonsingular and equal to zero at $r=0$. Asymptotically our solution behaves similar to the Reissner–Nördström-like solution in NGT.[80] Although asymptotically we see a Newtonian mass and an electric charge, at the origin ($r=0$) there is no mass or electric charge (only fermion charge l). Thus, it seems that we get "mass" without mass and "charge" without charge. The total charge for our solution is the same as the Coulomb charge (charge seen at infinity). The total mass is the same as the Newtonian mass (mass seen at infinity).

In this sense it could be treated as a kind of geon.[91] The Newtonian mass is the self-interaction energy of all fields. This is in the spirit of the Mach principle. The equality of the total and the Newtonian mass seems to be connected to the topological properties of the space-time.

For example, if we consider this solution as a model of the electron, we get a connection between the classical radius of the electron and its fermion number parameter l. Note that in general relativity the total energy associated with the electric field of a pointlike electron is infinite.

Our solution possesses a singularity at $r=0$ in the determinant of the full nonsymmetric metric. However, the (symmetric) metric seems to be less singular. There is no singularity for the function α and the determinant of the symmetric part is not zero. The function γ has a singularity only in the factor $1+l^4/r^4$ and the function $\omega = l^2/r^2$ has the usual singularity at $r=0$. The electric field is not singular. Our solution possesses one or two event horizons if the charge Q (and consequently the Newtonian mass) is sufficiently large. The solution seems to represent a bounded system of gravitational and electromagnetic fields. The radial energy density is zero at the origin, and finite everywhere. The metric is spatially flat at the origin. For a very small value of the parameter q (see Figure 7) the function $\alpha \simeq 1$, and $\gamma = 1 + l^4/r^4$. If the parameter q is equal to q_{electron}, one gets

$$1 \geq \alpha^{-1} = 1 - q^2_{\text{electron}} P(R) \geq 1 - q^2_{\text{electron}} \cdot P_{\max} \geq 1 - 10^{-74} \simeq 1 \qquad (22.1)$$

Thus, α is almost exactly one and γ is almost exactly $1 + l^4/r^4$. The metric is then as follows:

$$\mathbf{g}_{\mu\nu} = \begin{bmatrix} -1 & 0 & 0 & l^2/r^2 \\ 0 & -r^2 & 0 & 0 \\ 0 & 0 & -r^2 \sin^2\theta & 0 \\ -l^2/r^2 & 0 & 0 & 1 + l^4/r^4 \end{bmatrix} \qquad (22.2)$$

The symmetric part of this metric is spatially flat. It is easy to see that such behavior is valid for every elementary particle. The remarkable property of (22.2) is that it is described completely by the parameter l (fermion number), which plays the role of the second gravitational charge in the nonsymmetric theory of gravitation. It seems that the fermion number parameter should play a significant role in the unification of elementary particle theory and gravity. In equation (20.2) the fermion number parameter is much more important than the mass. Thus, the geometry of space-time on the level of elementary particles is determined by the second gravitational charge. The function α^{-1} in general relativity has the following form:

$$\alpha^{-1} = 1 - \frac{2m}{r} \tag{22.3}$$

This function describes the difference between the Schwarzschild solution and a Minkowski metric; in particular, the curvature of the space. In the solar system at the earth's orbit one finds

$$\alpha^{-1}(1\ \mathrm{au}) \cong 1 - 3 \times 10^{-8} \tag{22.4}$$

where $1\ \mathrm{au} = 1.45 \times 10^{8}$ km is one astronomical unit (the radius of the earth's orbit) and we have put into equation (22.3)

$$2m \simeq 5\ \mathrm{km} \tag{22.5}$$

which is the Schwarzschild radius of the sun. If we compare equation (22.4) with (22.1), we easily see that our solution with $q = q_{\mathrm{electron}}$ is spatially much flatter *everywhere* than three-space at the orbit of the earth.

Note that in equation (22.2) we get in a natural way the constant l, which has the dimension of length. Some authors claim that it is impossible to get a true unification of the gravitational field and elementary particles without a new universal constant of the dimension of length. In the nonsymmetric theory of gravitation there exists such a constant connected to the fermion number. The nonsymmetric Kaluza–Klein theory, which unifies the nonsymmetric theory of gravitation with a gauge field theory (i.e., the electromagnetic field), possesses this constant as well. This fact might enable this investigation to lead ultimately to a true unification of gravity and elementary particles.

Here are some prospects for further investigations:

1. Find nonstatic solutions if they exist.
2. Find axially symmetric stationary solutions of the field equations. This is more difficult, because there is no known axially symmetric stationary solution in the Einstein unified field theory and in NGT.
3. Extend our formalism to the non-Abelian, nonsymmetric Kaluza–Klein theory,[23,24] i.e., find such a solution for the case $G = SU(2)$ and $G = SU(2) \times U(1)$. This will offer a model of an electron or a lepton constructed from gravitational, electromagnetic, and weak interactions.
4. Extend our solution for the nonsymmetric Jordan–Thiry theory.[19]

Recently R. B. Mann[31] found eight classes of spherically symmetric and static solutions in NKKT. These solutions are more general and some of them have no singularities in gravitational and electromagnetic fields. Our solution is a special case of his solutions. Some of these solutions possess nonzero magnetic field ($B_0 \neq 0$) and nonzero $g_{[23]} = f \neq 0$. The nonsingular solutions are parametrized by fermion charge l^2, electric charge Q, and a new constant u_0. This constant is related to $g_{[23]}$ as l^2 is to $g_{[14]}$. It plays a similar role for $g_{[\mu\nu]}$ as the magnetic charge for $F_{\mu\nu}$. In some solutions the skewon singularity l^2/r^2 is replaced by an expression of the form $l^2/(r^4 + f^2)$.

Much work needs to be done to find the physical significance of these solutions. It is important to determine whether they are classically stable (for example, in the Poincaré sense). If they are, it would have very important consequences for the possible existence of quantum particles based on solitonlike solutions. Probably the nonsingular aspects of the solutions are a manifestation of the topological properties of NKKT. This is supported by the topological character of the current J_μ.

Finally, let us notice that our solution has many similarities with Demianski's[(90)] solution of coupled Born–Infeld and Einstein equations. This solution is nonsingular for a special choice of the integration constant ($c = 0$) as in our case. Moreover, it depends on the Born–Infeld constant and cannot be considered as a model of an electron. The mass of the solution is the self-interaction energy of the gravitational and electric fields. It seems that NKKT has many unexpected relations to nonlinear electrodynamics in curved space-time. This statement can also be supported by the form of the Born–Infeld Lagrangian. Originally[(87)] it was supposed that

$$\mathscr{L}_{\rm BI} = \frac{1}{4\pi} \{[-\det(b g_{\mu\nu})]^{1/2} - [-\det(b g_{\mu\nu} + F_{\mu\nu})]^{1/2}\} \qquad (22.6)$$

where $g_{\mu\nu} = g_{\nu\mu}$ is a symmetric metric tensor and $F_{\mu\nu}$ is the strength of the electromagnetic field, $b = \text{const}$. The form of the Lagrangian reveals its connection to the nonsymmetric field theory because of the nonsymmetric tensor

$$P_{\mu\nu} = b g_{\mu\nu} + F_{\mu\nu} \qquad (22.7)$$

However, in NGT and NKKT the skew-symmetric part of the metric has a different, i.e., gravitational interpretation.

APPENDIX

Using equations (2.9) and (2.11) from Ref. 82 and the equation

$$\frac{\omega^2}{\alpha\gamma - \omega^2} = \frac{l^4}{\beta^2 + f^2} \qquad (A.1)$$

one gets

$$A_{11}(\bar{\Gamma}) = -\tfrac{1}{2}\phi'' - \tfrac{1}{8}[(\phi') + 4C^2] + \frac{\alpha'}{4\alpha}\phi' + \frac{\omega^2}{8\gamma^2}[3(\dot{\phi})^2 + 4D^2]$$

$$+\left(\frac{\omega^2}{2\alpha\gamma}\phi' + \frac{\gamma'}{2\gamma}\right)\left(\frac{\alpha'}{2\alpha} - \frac{\omega^2}{2\alpha\gamma}\phi' - \frac{\gamma'}{2\gamma}\right)$$

$$-\frac{\partial}{\partial r}\left(\frac{\omega^2}{2\alpha\gamma} + \frac{\gamma'}{2\gamma}\right) + \frac{\partial}{\partial t}\left(\frac{\omega^2}{\gamma^2} + \frac{\dot{\alpha}}{2\gamma}\right)$$

$$+\left(\frac{\omega^2}{\gamma^2}\dot{\phi} + \frac{\dot{\alpha}}{2\gamma}\right)\left(\frac{\dot{\gamma}}{2\gamma} - \frac{\omega^2}{2\alpha\gamma}\dot{\phi} - \frac{\dot{\alpha}}{2\alpha} + \frac{1}{2}\dot{\phi}\right) \tag{A.2}$$

$$A_{44}(\bar{\Gamma}) = \tfrac{1}{2}\ddot{\phi} - \tfrac{1}{8}[(\dot{\phi})^2 + D^2] + \frac{\dot{\gamma}}{4\gamma}\dot{\phi} + \frac{\omega^2}{8\alpha^2}[3(\phi\ )^2 + 4C^2]$$

$$+\left(\frac{\omega^2}{2\alpha\gamma}\phi' + \frac{\dot{\alpha}}{2\alpha}\right)\left(\frac{\gamma'}{2\gamma} - \frac{\omega^2}{2\alpha\gamma}\phi' - \frac{\dot{\alpha}}{2\alpha}\right)$$

$$-\frac{\partial}{\partial t}\left(\frac{\omega^2}{2\alpha\gamma}\dot{\phi} + \frac{\dot{\alpha}}{2\alpha}\right) + \left(\frac{\alpha'}{2\alpha} - \frac{\omega^2}{2\alpha\gamma}\phi' - \frac{\gamma'}{2\gamma}\right)$$

$$+\frac{\partial}{\partial r}\left(\frac{\omega^2}{\alpha^2}\dot{\phi} + \frac{\gamma'}{2\alpha}\right) \tag{A.3}$$

$$A_{22}(\bar{\Gamma}) = \left\{\left(\frac{2fC - \beta\phi'}{4\alpha}\right)' + \frac{2fC - \beta\phi'}{8\alpha}\frac{\partial}{\partial r}\log[\omega^2(\beta^2 + f^2)]\right.$$

$$+\frac{B(f\phi' + 2\beta C)}{4\alpha} + 1 - \frac{\partial}{\partial t}\left(\frac{2fD - \beta\dot{\phi}}{4\gamma}\right)$$

$$\left.-\frac{2fD - \beta\dot{\phi}}{8\gamma}\frac{\partial}{\partial t}\log[\omega^2(\beta^2 + f^2)] - \frac{D}{4\gamma}(f\dot{\phi} + 2\beta D)\right\}$$

$$= \frac{1}{\sin^2\theta}A_{33}(\bar{\Gamma}) \tag{A.4}$$

$$A_{(14)}(\bar{\Gamma}) = \frac{\partial}{\partial r}\left(\frac{\omega^2}{4\alpha\gamma}\dot{\phi} + \frac{\dot{\alpha}}{4\alpha} - \frac{\dot{\gamma}}{4\gamma} - \frac{1}{4}\dot{\phi}'\right)$$

$$+\frac{\partial}{\partial t}\left(\frac{\omega^2}{4\alpha\gamma}\phi' + \frac{\gamma'}{4\gamma} - \frac{\alpha'}{4\alpha} - \frac{1}{4}\phi'\right)$$

$$+\tfrac{1}{2}\phi'\left(\frac{\omega^2}{2\alpha\gamma}\dot{\phi}+\frac{\dot{\alpha}}{2\alpha}-\frac{1}{4}\dot{\phi}\right)-\left(\frac{\omega^2}{\gamma^2}\dot{\phi}+\frac{\dot{\alpha}}{2\gamma}\right)\left(\frac{\omega^2}{2\alpha\gamma}\dot{\phi}+\frac{\dot{\alpha}}{2\alpha}\right)$$
$$+\left(\frac{\omega^2}{2\alpha\gamma}\phi'+\frac{\gamma'}{2\gamma}\right)\left(\frac{1}{2}\dot{\phi}+\frac{\omega^2}{2\alpha\gamma}\dot{\phi}+\frac{\dot{\alpha}}{2\alpha}\right)+\frac{\omega^2}{2\alpha\gamma}\phi'\dot{\phi}$$
$$-\frac{DC}{2l^2}\frac{\beta^2+f^2}{\alpha} \tag{A.5}$$

$$A_{[23]}(\bar{\Gamma})=\sin\theta\left[\left(\frac{f\phi'-2\beta C}{4\alpha}\right)'-\frac{C}{4\alpha}(2fC-\beta\phi')\right.$$
$$+\frac{1}{8\alpha}(f\phi'+2\beta C)\left(\frac{\alpha'}{\alpha}+\frac{\omega^2}{\alpha\gamma}\phi'+\frac{\gamma'}{\gamma}\right)$$
$$+\frac{1}{8\gamma}(f\dot{\phi}+2\beta D)\left(\frac{\dot{\gamma}}{\gamma}+\frac{\omega^2}{2\alpha\gamma}\dot{\phi}+\frac{\dot{\alpha}}{2\alpha}\right)$$
$$\left.-\frac{\partial}{\partial t}\left(\frac{f\dot{\phi}+2\beta D}{4\gamma}\right)+\frac{D}{4}(2fD-\beta\dot{\phi})\right] \tag{A.6}$$

where

$$\phi=\log(\beta^2+f^2)$$
$$C=\frac{f\beta'-\beta f'}{\beta^2+f^2}$$
$$D=\frac{\dot{\beta}f-\beta\dot{f}}{\beta^2+f^2}$$

An overdot means a derivative with respect to time t, and a prime means a derivative with respect to radius r. We also have

$$A_{[14]}(\bar{\Gamma})=\frac{\omega}{8\alpha}[(\phi')^2+4C^2]-\frac{\omega}{8\gamma}[(\dot{\phi}')^2+4D^2]$$
$$+\frac{\omega}{4\alpha}\phi'(\phi'+\dot{\phi})-\frac{1}{2}\frac{\partial}{\partial t}\left(\dot{\phi}\frac{\omega}{\gamma}\right)$$

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